Chapter 9

Analytic Continuation

For every complex problem, there is a solution that is simple, neat, and wrong.

- H. L. Mencken

9.1 Analytic Continuation

Suppose there is a function, $f_1(z)$ that is analytic in the domain $D_1$ and another analytic function, $f_2(z)$ that is analytic in the domain $D_2$. (See Figure 9.1.)

If the two domains overlap and $f_1(z) = f_2(z)$ in the overlap region $D_1 \cap D_2$, then $f_2(z)$ is called an analytic continuation of $f_1(z)$. This is an appropriate name since $f_2(z)$ continues the definition of $f_1(z)$ outside of its original domain of definition $D_1$. We can define a function $f(z)$ that is analytic in the union of the domains $D_1 \cup D_2$. On the domain $D_1$ we have $f(z) = f_1(z)$ and $f(z) = f_2(z)$ on $D_2$. $f_1(z)$ and $f_2(z)$ are called function elements. There is an analytic continuation even if the two domains only share an arc and not a two dimensional region.

With more overlapping domains $D_3, D_4, \ldots$ we could perhaps extend $f_1(z)$ to more of the complex plane. Sometimes it is impossible to extend a function beyond the boundary of a domain. This is known as a natural boundary. If a
function $f_1(z)$ is analytically continued to a domain $D_n$ along two different paths, (See Figure 9.2.), then the two analytic continuations are identical as long as the paths do not enclose a branch point of the function. This is the uniqueness theorem of analytic continuation.

Consider an analytic function $f(z)$ defined in the domain $D$. Suppose that $f(z) = 0$ on the arc $AB$, (see Figure 9.3.) Then $f(z) = 0$ in all of $D$.

Consider a point $\zeta$ on $AB$. The Taylor series expansion of $f(z)$ about the point $z = \zeta$ converges in a circle $C$ at
least up to the boundary of $D$. The derivative of $f(z)$ at the point $z = \zeta$ is

$$f'(\zeta) = \lim_{\Delta z \to 0} \frac{f(\zeta + \Delta z) - f(\zeta)}{\Delta z}$$

If $\Delta z$ is in the direction of the arc, then $f'(\zeta)$ vanishes as well as all higher derivatives, $f'(\zeta) = f''(\zeta) = f'''(\zeta) = \cdots = 0$. Thus we see that $f(z) = 0$ inside $C$. By taking Taylor series expansions about points on $AB$ or inside of $C$ we see that $f(z) = 0$ in $D$.

**Result 9.1.1** Let $f_1(z)$ and $f_2(z)$ be analytic functions defined in $D$. If $f_1(z) = f_2(z)$ for the points in a region or on an arc in $D$, then $f_1(z) = f_2(z)$ for all points in $D$.

To prove Result 9.1.1, we define the analytic function $g(z) = f_1(z) - f_2(z)$. Since $g(z)$ vanishes in the region or on the arc, then $g(z) = 0$ and hence $f_1(z) = f_2(z)$ for all points in $D$. 

439
Consider analytic functions \( f_1(z) \) and \( f_2(z) \) defined on the domains \( D_1 \) and \( D_2 \), respectively. Suppose that \( D_1 \cap D_2 \) is a region or an arc and that \( f_1(z) = f_2(z) \) for all \( z \in D_1 \cap D_2 \). (See Figure 9.4.) Then the function

\[
f(z) = \begin{cases} 
  f_1(z) & \text{for } z \in D_1, \\
  f_2(z) & \text{for } z \in D_2,
\end{cases}
\]

is analytic in \( D_1 \cup D_2 \).

**Figure 9.4:** Domains that Intersect in a Region or an Arc

Result 9.1.2 follows directly from Result 9.1.1.

### 9.2 Analytic Continuation of Sums

**Example 9.2.1** Consider the function

\[
f_1(z) = \sum_{n=0}^{\infty} z^n.
\]

The sum converges uniformly for \( D_1 = |z| \leq r < 1 \). Since the derivative also converges in this domain, the function is analytic there.
Now consider the function

\[ f_2(z) = \frac{1}{1 - z}. \]

This function is analytic everywhere except the point \( z = 1 \). On the domain \( D_1 \),

\[ f_2(z) = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = f_1(z) \]

Analytic continuation tells us that there is a function that is analytic on the union of the two domains. Here, the domain is the entire \( z \) plane except the point \( z = 1 \) and the function is

\[ f(z) = \frac{1}{1 - z}. \]

\( \frac{1}{1 - z} \) is said to be an analytic continuation of \( \sum_{n=0}^{\infty} z^n \).
9.3 Analytic Functions Defined in Terms of Real Variables

Result 9.3.1 An analytic function, \( u(x, y) + iv(x, y) \) can be written in terms of a function of a complex variable, \( f(z) = u(x, y) + iv(x, y) \).

Result 9.3.1 is proved in Exercise 9.1.

Example 9.3.1

\[
f(z) = \cosh y \sin x (x e^x \cos y - y e^x \sin y) - \cos x \sinh y (y e^x \cos y + x e^x \sin y) \\
+ i \left[ \cosh y \sin x (y e^x \cos y + x e^x \sin y) + \cos x \sinh y (x e^x \cos y - y e^x \sin y) \right]
\]

is an analytic function. Express \( f(z) \) in terms of \( z \).

On the real line, \( y = 0 \), \( f(z) \) is

\[
f(z = x) = x e^x \sin x
\]

(Recall that \( \cos(0) = \cosh(0) = 1 \) and \( \sin(0) = \sinh(0) = 0 \).)

The analytic continuation of \( f(z) \) into the complex plane is

\[
f(z) = z e^z \sin z.
\]

Alternatively, for \( x = 0 \) we have

\[
f(z = iy) = y \sinh y (\cos y - i \sin y).
\]

The analytic continuation from the imaginary axis to the complex plane is

\[
f(z) = -iz \sinh(-iz) (\cos(-iz) - i \sin(-iz)) \\
= iz \sinh(iz) (\cos(iz) + i \sin(iz)) \\
= z \sin z e^z.
\]
Example 9.3.2 Consider \( u = e^{-x}(x \sin y - y \cos y) \). Find \( v \) such that \( f(z) = u + iv \) is analytic. From the Cauchy-Riemann equations,

\[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y
\]

\[
\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x} \cos y - x e^{-x} \cos y - y e^{-x} \sin y
\]

Integrate the first equation with respect to \( y \).

\[
v = -e^{-x} \cos y + x e^{-x} \cos y + e^{-x}(y \sin y + \cos y) + F(x)
\]

\[
v = y e^{-x} \sin y + x e^{-x} \cos y + F(x)
\]

\( F(x) \) is an arbitrary function of \( x \). Substitute this expression for \( v \) into the equation for \( \partial v / \partial x \).

\[
-y e^{-x} \sin y - x e^{-x} \cos y + e^{-x} \cos y + F'(x) = -y e^{-x} \sin y - x e^{-x} \cos y + e^{-x} \cos y
\]

Thus \( F'(x) = 0 \) and \( F(x) = c \).

\[
v = e^{-x}(y \sin y + x \cos y) + c
\]

Example 9.3.3 Find \( f(z) \) in the previous example. (Up to the additive constant.)

Method 1

\[
f(z) = u + w
\]

\[
= e^{-x}(x \sin y - y \cos y) + i e^{-x}(y \sin y + x \cos y)
\]

\[
= e^{-x} \left\{ x \left( \frac{e^{iy} - e^{-iy}}{i 2} \right) - y \left( \frac{e^{iy} + e^{-iy}}{2} \right) \right\} + i e^{-x} \left\{ y \left( \frac{e^{iy} - e^{-iy}}{i 2} \right) + x \left( \frac{e^{iy} + e^{-iy}}{2} \right) \right\}
\]

\[
= i(x + iy) e^{-x} + y e^{-x}
\]

\[
= i e^{-x} z
\]
Method 2 \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is an analytic function.

On the real axis, \( y = 0 \), \( f(z) \) is
\[
f(z = x) = u(x, 0) + iv(x, 0)
= e^{-x}(x \sin 0 - 0 \cos 0) + iv^{-x}(0 \sin 0 + x \cos 0)
= ix e^{-x}
\]

Suppose there is an analytic continuation of \( f(z) \) into the complex plane. If such a continuation, \( f(z) \), exists, then it must be equal to \( f(z = x) \) on the real axis. An obvious choice for the analytic continuation is
\[
f(z) = u(z, 0) + iv(z, 0)
\]
since this is clearly equal to \( u(x, 0) + iv(x, 0) \) when \( z \) is real. Thus we obtain
\[
f(z) = x e^{-z}
\]

Example 9.3.4 Consider \( f(z) = u(x, y) + iv(x, y) \). Show that \( f'(z) = u_x(z, 0) - iu_y(z, 0) \).

\[
f'(z) = u_x + iv_x
= u_x - iu_y
\]

\( f'(z) \) is an analytic function. On the real axis, \( z = x \), \( f'(z) \) is
\[
f'(z = x) = u_x(x, 0) - iu_y(x, 0)
\]

Now \( f'(z = x) \) is defined on the real line. An analytic continuation of \( f'(z = x) \) into the complex plane is
\[
\boxed{f'(z) = u_x(z, 0) - iu_y(z, 0)}
\]

Example 9.3.5 Again consider the problem of finding \( f(z) \) given that \( u(x, y) = e^{-x}(x \sin y - y \cos y) \). Now we can use the result of the previous example to do this problem.

\[
u_x(x, y) = \frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y
\]
\[
u_y(x, y) = \frac{\partial u}{\partial y} = x e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y
\]
\[ f'(z) = u_x(z, 0) - u_y(z, 0) \]
\[ = 0 - i \left( z e^{-z} - e^{-z} \right) \]
\[ = i \left( -z e^{-z} + e^{-z} \right) \]

Integration yields the result

\[ f(z) = i z e^{-z} + c \]

**Example 9.3.6** Find \( f(z) \) given that

\[ u(x, y) = \cos x \cosh^2 y \sin x + \cos x \sin x \sinh^2 y \]
\[ v(x, y) = \cos^2 x \cosh y \sinh y - \cosh y \sin^2 x \sinh y \]

\[ f(z) = u(x, y) + w(x, y) \] is an analytic function. On the real line, \( f(z) \) is

\[ f(z = x) = u(x, 0) + w(x, 0) \]
\[ = \cos x \cosh^2 0 \sin x + \cos x \sin x \sinh^2 0 + i \left( \cos^2 x \cosh 0 \sinh 0 - \cosh 0 \sin^2 x \sinh 0 \right) \]
\[ = \cos x \sin x \]

Now we know the definition of \( f(z) \) on the real line. We would like to find an analytic continuation of \( f(z) \) into the complex plane. An obvious choice for \( f(z) \) is

\[ f(z) = \cos z \sin z \]

Using trig identities we can write this as

\[ f(z) = \frac{\sin(2z)}{2} \]

**Example 9.3.7** Find \( f(z) \) given only that

\[ u(x, y) = \cos x \cosh^2 y \sin x + \cos x \sin x \sinh^2 y. \]
Recall that

\[ f'(z) = u_x + iv_x \]
\[ = u_x - iv_y \]

Differentiating \( u(x, y) \),

\[ u_x = \cos^2 x \cosh^2 y - \cosh^2 y \sin^2 x + \cos^2 x \sinh^2 y - \sin^2 x \sinh^2 y \]
\[ u_y = 4 \cos x \cosh y \sin x \sinh y \]

\( f'(z) \) is an analytic function. On the real axis, \( f'(z) \) is

\[ f'(z = x) = \cos^2 x - \sin^2 x \]

Using trig identities we can write this as

\[ f'(z = x) = \cos(2x) \]

Now we find an analytic continuation of \( f'(z = x) \) into the complex plane.

\[ f'(z) = \cos(2z) \]

Integration yields the result

\[ f(z) = \frac{\sin(2z)}{2} + c \]

### 9.3.1 Polar Coordinates

**Example 9.3.8** Is

\[ u(r, \theta) = r(\log r \cos \theta - \theta \sin \theta) \]

the real part of an analytic function?
The Laplacian in polar coordinates is

\[ \Delta \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}. \]

We calculate the partial derivatives of \( u \).

\[
\begin{align*}
\frac{\partial u}{\partial r} &= \cos \theta + \log r \cos \theta - \theta \sin \theta \\
\frac{\partial u}{\partial \theta} &= r \cos \theta + r \log r \cos \theta - r \theta \sin \theta \\
\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) &= 2 \cos \theta + \log r \cos \theta - \theta \sin \theta \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) &= \frac{1}{r} (2 \cos \theta + \log r \cos \theta - \theta \sin \theta) \\
\frac{\partial u}{\partial \theta} &= -r (\theta \cos \theta + \sin \theta + \log r \sin \theta) \\
\frac{\partial^2 u}{\partial \theta^2} &= r (-2 \cos \theta - \log r \cos \theta + \theta \sin \theta) \\
\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= \frac{1}{r} (-2 \cos \theta - \log r \cos \theta + \theta \sin \theta)
\end{align*}
\]

From the above we see that

\[ \Delta u \]

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \]

Therefore \( u \) is harmonic and is the real part of some analytic function.

**Example 9.3.9** Find an analytic function \( f(z) \) whose real part is

\[ u(r, \theta) = r (\log r \cos \theta - \theta \sin \theta). \]
Let $f(z) = u(r, \theta) + iv(r, \theta)$. The Cauchy-Riemann equations are

$$u_r = \frac{v_\theta}{r}, \quad u_\theta = -rv_r.$$ 

Using the partial derivatives in the above example, we obtain two partial differential equations for $v(r, \theta)$.

$$v_r = -\frac{u_\theta}{r} = \theta \cos \theta + \sin \theta + \log r \sin \theta$$
$$v_\theta = ru_r = r (\cos \theta + \log r \cos \theta - \theta \sin \theta)$$

Integrating the equation for $v_\theta$ yields

$$v = r (\theta \cos \theta + \log r \sin \theta) + F(r)$$

where $F(r)$ is a constant of integration.

Substituting our expression for $v$ into the equation for $v_r$ yields

$$\theta \cos \theta + \log r \sin \theta + \sin \theta + F'(r) = \theta \cos \theta + \sin \theta + \log r \sin \theta$$
$$F'(r) = 0$$
$$F(r) = \text{const}$$

Thus we see that

$$f(z) = u + iv$$
$$= r (\log r \cos \theta - \theta \sin \theta) + vr (\theta \cos \theta + \log r \sin \theta) + \text{const}$$

$f(z)$ is an analytic function. On the line $\theta = 0$, $f(z)$ is

$$f(z = r) = r (\log r) + vr(0) + \text{const}$$
$$= r \log r + \text{const}$$

The analytic continuation into the complex plane is

$$f(z) = z \log z + \text{const}$$
Example 9.3.10  Find the formula in polar coordinates that is analogous to

\[ f'(z) = u_x(z, 0) - iu_y(z, 0). \]

We know that

\[ \frac{df}{dz} = e^{-i\theta} \frac{\partial f}{\partial r}. \]

If \( f(z) = u(r, \theta) + iv(r, \theta) \) then

\[ \frac{df}{dz} = e^{-i\theta} (u_r + iv_r) \]

From the Cauchy-Riemann equations, we have \( v_r = -u_\theta / r \).

\[ \frac{df}{dz} = e^{-i\theta} \left( u_r - \frac{i\ u_\theta}{r} \right) \]

\( f'(z) \) is an analytic function. On the line \( \theta = 0 \), \( f(z) \) is

\[ f'(z = r) = u_r(r, 0) - \frac{i\ u_\theta(r, 0)}{r} \]

The analytic continuation of \( f'(z) \) into the complex plane is

\[ f'(z) = u_r(z, 0) - \frac{i\ u_\theta(z, 0)}{r}. \]

Example 9.3.11  Find an analytic function \( f(z) \) whose real part is

\[ u(r, \theta) = r (\log r \cos \theta - \theta \sin \theta). \]

\[ u_r(r, \theta) = (\log r \cos \theta - \theta \sin \theta) + \cos \theta \]

\[ u_\theta(r, \theta) = r (-\log r \sin \theta - \sin \theta - \theta \cos \theta) \]

449
\[ f'(z) = u_r(z, 0) - \frac{i}{r} u_\theta(z, 0) \]
\[ = \log z + 1 \]

*Integrating* \( f'(z) \) *yields*

\[ f(z) = z \log z + i\cdot. \]

### 9.3.2 Analytic Functions Defined in Terms of Their Real or Imaginary Parts

Consider an analytic function: \( f(z) = u(x, y) + iv(x, y) \). We differentiate this expression.

\[ f'(z) = u_x(x, y) + iv_x(x, y) \]

We apply the Cauchy-Riemann equation \( v_x = -u_y \).

\[ f'(z) = u_x(x, y) - iv_y(x, y). \quad (9.1) \]

Now consider the function of a complex variable, \( g(\zeta) \):

\[ g(\zeta) = u_x(x, \zeta) - iv_y(x, \zeta) = u_x(x, \xi + i\psi) - iv_y(x, \xi + i\psi). \]

This function is analytic where \( f(\zeta) \) is analytic. To show this we first verify that the derivatives in the \( \xi \) and \( \psi \) directions are equal.

\[ \frac{\partial}{\partial \xi} g(\zeta) = u_{xy}(x, \xi + i\psi) - iv_{yy}(x, \xi + i\psi) \]

\[ -i \frac{\partial}{\partial \psi} g(\zeta) = -i (u_{xy}(x, \xi + i\psi) + u_{yy}(x, \xi + i\psi)) = u_{xy}(x, \xi + i\psi) - iv_{yy}(x, \xi + i\psi) \]

Since these partial derivatives are equal and continuous, \( g(\zeta) \) is analytic. We evaluate the function \( g(\zeta) \) at \( \zeta = -ix \).

(Substitute \( y = -ix \) into Equation 9.1.)

\[ f'(2x) = u_x(x, -ix) - iv_y(x, -ix) \]

450
We make a change of variables to solve for $f'(x)$.

$$f'(x) = u_x \left( \frac{x}{2}, -\frac{i}{2} \right) - i u_y \left( \frac{x}{2}, -\frac{i}{2} \right).$$

If the expression is non-singular, then this defines the analytic function, $f'(z)$, on the real axis. The analytic continuation to the complex plane is

$$f'(z) = u_x \left( \frac{z}{2}, -\frac{i}{2} \right) - i u_y \left( \frac{z}{2}, -\frac{i}{2} \right).$$

Note that $\frac{d}{dz} 2u(z/2, -iz/2) = u_x(z/2, -iz/2) - iu_y(z/2, -iz/2)$. We integrate the equation to obtain:

$$f(z) = 2u \left( \frac{z}{2}, -\frac{i}{2} \right) + c.$$

We know that the real part of an analytic function determines that function to within an additive constant. Assuming that the above expression is non-singular, we have found a formula for writing an analytic function in terms of its real part. With the same method, we can find how to write an analytic function in terms of its imaginary part, $v$.

We can also derive formulas if $u$ and $v$ are expressed in polar coordinates:

$$f(z) = u(r, \theta) + iv(r, \theta).$$
**Result 9.3.2** If \( f(z) = u(x, y) + iv(x, y) \) is analytic and the expressions are non-singular, then

\[
f(z) = 2u \left( \frac{z}{2}, -\frac{-i \bar{z}}{2} \right) + \text{const} \quad (9.2)
\]

\[
f(z) = i2v \left( \frac{z}{2}, -\frac{i \bar{z}}{2} \right) + \text{const}. \quad (9.3)
\]

If \( f(z) = u(r, \theta) + iv(r, \theta) \) is analytic and the expressions are non-singular, then

\[
f(z) = 2u \left( z^{1/2}, -\frac{i}{2} \log z \right) + \text{const} \quad (9.4)
\]

\[
f(z) = i2v \left( z^{1/2}, -\frac{i}{2} \log z \right) + \text{const}. \quad (9.5)
\]

**Example 9.3.12** Consider the problem of finding \( f(z) \) given that \( u(x, y) = e^{-x}(x \sin y - y \cos y) \).

\[
f(z) = 2u \left( \frac{z}{2}, -\frac{i \bar{z}}{2} \right)
\]

\[
= 2e^{-z/2} \left( \frac{z}{2} \sin \left( -\frac{i \bar{z}}{2} \right) + \frac{i \bar{z}}{2} \cos \left( -\frac{i \bar{z}}{2} \right) \right) + c
\]

\[
= iz e^{-z/2} \left( i \sin \left( \frac{i \bar{z}}{2} \right) + \cos \left( -\frac{i \bar{z}}{2} \right) \right) + c
\]

\[
= iz e^{-z/2} (e^{-z/2}) + c
\]

\[
= iz e^{-z} + c
\]

**Example 9.3.13** Consider

\[
\text{Log } z = \frac{1}{2} \text{Log } (x^2 + y^2) + i \text{Arctan}(x, y).
\]
We try to construct the analytic function from its real part using Equation 9.2.

\[ f(z) = 2u \left( \frac{z}{2}, -\frac{iz}{2} \right) + c \]
\[ = 2 \frac{1}{2} \log \left( \left( \frac{z}{2} \right)^2 + \left( -\frac{iz}{2} \right)^2 \right) + c \]
\[ = \log(0) + c \]

We obtain a singular expression, so the method fails.

**Example 9.3.14** Again consider the logarithm, this time written in terms of polar coordinates.

\[ \log z = \log r + i\theta \]

We try to construct the analytic function from its real part using Equation 9.4.

\[ f(z) = 2u \left( z^{1/2}, -\frac{i}{2} \log z \right) + c \]
\[ = 2 \log \left( z^{1/2} \right) + c \]
\[ = \log z + c \]

With this method we recover the analytic function.