Chapter 8

Analytic Functions

Students need encouragement. So if a student gets an answer right, tell them it was a lucky guess. That way, they develop a good, lucky feeling.\textsuperscript{1}

-Jack Handey

8.1 Complex Derivatives

Functions of a Real Variable. The derivative of a function of a real variable is

\[
\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.
\]

If the limit exists then the function is differentiable at the point \( x \). Note that \( \Delta x \) can approach zero from above or below. The limit cannot depend on the direction in which \( \Delta x \) vanishes.

Consider \( f(x) = |x| \). The function is not differentiable at \( x = 0 \) since

\[
\lim_{\Delta x \to 0^+} |0 + \Delta x| - |0| \over \Delta x = 1
\]

\textsuperscript{1}Quote slightly modified.
and

$$\lim_{\Delta x \to 0^-} \frac{|0 + \Delta x| - |0|}{\Delta x} = -1.$$ 

**Analyticity.** The complex derivative, (or simply derivative if the context is clear), is defined,

$$\frac{d}{dz}f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$ 

The complex derivative exists if this limit exists. This means that the value of the limit is independent of the manner in which $\Delta z \to 0$. If the complex derivative exists at a point, then we say that the function is complex differentiable there.

A function of a complex variable is analytic at a point $z_0$ if the complex derivative exists in a neighborhood about that point. The function is analytic in an open set if it has a complex derivative at each point in that set. Note that complex differentiable has a different meaning than analytic. Analyticity refers to the behavior of a function on an open set. A function can be complex differentiable at isolated points, but the function would not be analytic at those points. Analytic functions are also called regular or holomorphic. If a function is analytic everywhere in the finite complex plane, it is called entire.

**Example 8.1.1** Consider $z^n$, $n \in \mathbb{Z}^+$, Is the function differentiable? Is it analytic? What is the value of the derivative?

We determine differentiability by trying to differentiate the function. We use the limit definition of differentiation. We will use Newton’s binomial formula to expand $(z + \Delta z)^n$.

$$\frac{d}{dz}z^n = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^n + nz^{n-1}\Delta z + \frac{n(n-1)}{2}z^{n-2}\Delta z^2 + \cdots + \Delta z^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left( nz^{n-1} + \frac{n(n-1)}{2}z^{n-2}\Delta z + \cdots + \Delta z^{n-1} \right)$$

$$= nz^{n-1}.$$
The derivative exists everywhere. The function is analytic in the whole complex plane so it is entire. The value of the derivative is \( \frac{d}{dz} = nz^{n-1} \).

**Example 8.1.2** We will show that \( f(z) = \bar{z} \) is not differentiable. Consider its derivative.

\[
\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.
\]

\[
\frac{d}{dz} \bar{z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \Delta z - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1
\]

First we take \( \Delta z = \Delta x \) and evaluate the limit.

\[
\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1
\]

Then we take \( \Delta z = i \Delta y \).

\[
\lim_{\Delta y \to 0} \frac{-i \Delta y}{\Delta y} = -1
\]

Since the limit depends on the way that \( \Delta z \to 0 \), the function is nowhere differentiable. Thus the function is not analytic.

**Complex Derivatives in Terms of Plane Coordinates.** Let \( z = \zeta(\xi, \psi) \) be a system of coordinates in the complex plane. (For example, we could have Cartesian coordinates \( z = \zeta(x, y) = x + iy \) or polar coordinates \( z = \zeta(r, \theta) = re^{i\theta} \). Let \( f(z) = \phi(\xi, \psi) \) be a complex-valued function. (For example we might have a function in the form \( \phi(x, y) = u(x, y) + iv(x, y) \) or \( \phi(r, \theta) = R(r, \theta)e^{i\Theta(r, \theta)} \).) If \( f(z) = \phi(\xi, \psi) \) is analytic, its complex derivative is
equal to the derivative in any direction. In particular, it is equal to the derivatives in the coordinate directions.

\[
\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\phi(\xi + \Delta \xi, \psi) - \phi(\xi, \psi)}{\Delta \xi} = \left( \frac{\partial \phi}{\partial \xi} \right)^{-1} \frac{\partial \phi}{\partial \xi}
\]

\[
\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta \psi \to 0} \frac{\phi(\xi, \psi + \Delta \psi) - \phi(\xi, \psi)}{\Delta \psi} = \left( \frac{\partial \phi}{\partial \psi} \right)^{-1} \frac{\partial \phi}{\partial \psi}
\]

**Example 8.1.3** Consider the Cartesian coordinates \( z = x + iy \). We write the complex derivative as derivatives in the coordinate directions for \( f(z) = \phi(x, y) \).

\[
\frac{df}{dz} = \left( \frac{\partial (x + iy)}{\partial x} \right)^{-1} \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x}
\]

\[
\frac{df}{dz} = \left( \frac{\partial (x + iy)}{\partial y} \right)^{-1} \frac{\partial \phi}{\partial y} = -i \frac{\partial \phi}{\partial y}
\]

We write this in operator notation.

\[
\frac{d}{dz} = \frac{\partial}{\partial x} = -i \frac{\partial}{\partial y}
\]

**Example 8.1.4** In Example 8.1.1 we showed that \( z^n, n \in \mathbb{Z}^+ \), is an entire function and that \( \frac{d}{dz} z^n = nz^{n-1} \). Now we corroborate this by calculating the complex derivative in the Cartesian coordinate directions.

\[
\frac{d}{dz} z^n = \frac{\partial}{\partial x} (x + iy)^n
\]

\[= n(x + iy)^{n-1}\]

\[= nz^{n-1}\]

\[
\frac{d}{dz} z^n = -i \frac{\partial}{\partial y} (x + iy)^n
\]

\[= -in(x + iy)^{n-1}\]

\[= nz^{n-1}\]
Complex Derivatives are Not the Same as Partial Derivatives  
Recall from calculus that

\[ f(x, y) = g(s, t) \rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial g}{\partial t} \frac{\partial t}{\partial x} \]

Do not make the mistake of using a similar formula for functions of a complex variable. If \( f(z) = \phi(x, y) \) then

\[ \frac{df}{dz} \neq \frac{\partial \phi}{\partial x} \frac{dx}{dz} + \frac{\partial \phi}{\partial y} \frac{dy}{dz}. \]

This is because the \( \frac{d}{dz} \) operator means “The derivative in any direction in the complex plane.” Since \( f(z) \) is analytic, \( f'(z) \) is the same no matter in which direction we take the derivative.

Rules of Differentiation. For an analytic function defined in terms of \( z \) we can calculate the complex derivative using all the usual rules of differentiation that we know from calculus like the product rule,

\[ \frac{d}{dz} f(z)g(z) = f'(z)g(z) + f(z)g'(z), \]

or the chain rule,

\[ \frac{d}{dz} f(g(z)) = f'(g(z))g'(z). \]

This is because the complex derivative derives its properties from properties of limits, just like its real variable counterpart.
Result 8.1.1 The complex derivative is,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$ 

The complex derivative is defined if the limit exists and is independent of the manner in which \(\Delta z \to 0\). A function is analytic at a point if the complex derivative exists in a neighborhood of that point.

Let \(z = \zeta(\xi, \psi)\) define coordinates in the complex plane. The complex derivative in the coordinate directions is

$$\frac{d}{dz} = \left(\frac{\partial \zeta}{\partial \xi}\right)^{-1} \frac{\partial}{\partial \xi} = \left(\frac{\partial \zeta}{\partial \psi}\right)^{-1} \frac{\partial}{\partial \psi}.$$ 

In Cartesian coordinates, this is

$$\frac{d}{dz} = \frac{\partial}{\partial x} = -i \frac{\partial}{\partial y}.$$ 

In polar coordinates, this is

$$\frac{d}{dz} = e^{-i\theta} \frac{\partial}{\partial r} = -\frac{i}{r} e^{-i\theta} \frac{\partial}{\partial \theta}.$$ 

Since the complex derivative is defined with the same limit formula as real derivatives, all the rules from the calculus of functions of a real variable may be used to differentiate functions of a complex variable.

Example 8.1.5 We have shown that \(z^n, n \in \mathbb{Z}^+\), is an entire function. Now we corroborate that \(\frac{d}{dz} z^n = n z^{n-1}\) by
calculating the complex derivative in the polar coordinate directions.

\[
\frac{d}{dz} z^n = e^{-i\theta} \frac{\partial}{\partial r} r^n e^{in\theta} \\
= e^{-i\theta} nr^{n-1} e^{in\theta} \\
= nr^{n-1} e^{i(n-1)\theta} \\
= nz^{n-1}
\]

\[
\frac{d}{dz} z^n = -\frac{i}{r} e^{-i\theta} \frac{\partial}{\partial \theta} r^n e^{in\theta} \\
= -\frac{i}{r} e^{-i\theta} r^{n-1} e^{in\theta} \\
= nr^{n-1} e^{i(n-1)\theta} \\
= nz^{n-1}
\]

**Analytic Functions can be Written in Terms of z.** Consider an analytic function expressed in terms of \( x \) and \( y \), \( \phi(x, y) \). We can write \( \phi \) as a function of \( z = x + iy \) and \( \overline{z} = x - iy \).

\[
f(z, \overline{z}) = \phi \left( \frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i} \right)
\]

We treat \( z \) and \( \overline{z} \) as independent variables. We find the partial derivatives with respect to these variables.

\[
\frac{\partial}{\partial z} = \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial}{\partial y} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\
\frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}} \frac{\partial}{\partial y} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

Since \( \phi \) is analytic, the complex derivatives in the \( x \) and \( y \) directions are equal.

\[
\frac{\partial \phi}{\partial x} = -i \frac{\partial \phi}{\partial y}
\]
The partial derivative of \( f(z, \bar{z}) \) with respect to \( \bar{z} \) is zero.

\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right) = 0
\]

Thus \( f(z, \bar{z}) \) has no functional dependence on \( \bar{z} \), it can be written as a function of \( z \) alone.

If we were considering an analytic function expressed in polar coordinates \( \phi(r, \theta) \), then we could write it in Cartesian coordinates with the substitutions:

\[
r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(x, y).
\]

Thus we could write \( \phi(r, \theta) \) as a function of \( z \) alone.

**Result 8.1.2** Any analytic function \( \phi(x, y) \) or \( \phi(r, \theta) \) can be written as a function of \( z \) alone.

### 8.2 Cauchy-Riemann Equations

If we know that a function is analytic, then we have a convenient way of determining its complex derivative. We just express the complex derivative in terms of the derivative in a coordinate direction. However, we don’t have a nice way of determining if a function is analytic. The definition of complex derivative in terms of a limit is cumbersome to work with. In this section we remedy this problem.

**A necessary condition for analyticity.** Consider a function \( f(z) = \phi(x, y) \). If \( f(z) \) is analytic, the complex derivative is equal to the derivatives in the coordinate directions. We equate the derivatives in the \( x \) and \( y \) directions to obtain the **Cauchy-Riemann equations** in Cartesian coordinates.

\[
\phi_x = -\imath \phi_y
\]

This equation is a necessary condition for the analyticity of \( f(z) \).

Let \( \phi(x, y) = u(x, y) + \imath v(x, y) \) where \( u \) and \( v \) are real-valued functions. We equate the real and imaginary parts of Equation 8.1 to obtain another form for the Cauchy-Riemann equations in Cartesian coordinates.

\[
\begin{align*}
u_x &= v_y, \\
u_y &= -v_x.
\end{align*}
\]
Note that this is a necessary and not a sufficient condition for analyticity of \( f(z) \). That is, \( u \) and \( v \) may satisfy the Cauchy-Riemann equations but \( f(z) \) may not be analytic. At this point, Cauchy-Riemann equations give us an easy test for determining if a function is not analytic.

**Example 8.2.1** In Example 8.1.2 we showed that \( \overline{z} \) is not analytic using the definition of complex differentiation. Now we obtain the same result using the Cauchy-Riemann equations.

\[
\overline{z} = x - iy \\
u_x = 1, \quad v_y = -1
\]

We see that the first Cauchy-Riemann equation is not satisfied; the function is not analytic at any point.

**A sufficient condition for analyticity.** A sufficient condition for \( f(z) = \phi(x, y) \) to be analytic at a point \( z_0 = (x_0, y_0) \) is that the partial derivatives of \( \phi(x, y) \) exist and are continuous in some neighborhood of \( z_0 \) and satisfy the Cauchy-Riemann equations there. If the partial derivatives of \( \phi \) exist and are continuous then

\[
\phi(x + \Delta x, y + \Delta y) = \phi(x, y) + \Delta x \phi_x(x, y) + \Delta y \phi_y(x, y) + o(\Delta x) + o(\Delta y).
\]

Here the notation \( o(\Delta x) \) means “terms smaller than \( \Delta x \)”. We calculate the derivative of \( f(z) \).

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
= \lim_{\Delta x, \Delta y \to 0} \frac{\phi(x + \Delta x, y + \Delta y) - \phi(x, y)}{\Delta x + i\Delta y} \\
= \lim_{\Delta x, \Delta y \to 0} \frac{\phi(x, y) + \Delta x \phi_x(x, y) + \Delta y \phi_y(x, y) + o(\Delta x) + o(\Delta y) - \phi(x, y)}{\Delta x + i\Delta y} \\
= \lim_{\Delta x, \Delta y \to 0} \frac{\Delta x \phi_x(x, y) + \Delta y \phi_y(x, y) + o(\Delta x) + o(\Delta y)}{\Delta x + i\Delta y}
\]
Here we use the Cauchy-Riemann equations.

\[
\begin{align*}
&= \lim_{\Delta x, \Delta y \to 0} \frac{\Delta x + i\Delta y}{\Delta x + i\Delta y} \phi_x(x, y) + \lim_{\Delta x, \Delta y \to 0} \frac{o(\Delta x) + o(\Delta y)}{\Delta x + i\Delta y} \\
&= \phi_x(x, y)
\end{align*}
\]

Thus we see that the derivative is well defined.

**Cauchy-Riemann Equations in General Coordinates**  Let \( z = \zeta(\xi, \psi) \) be a system of coordinates in the complex plane. Let \( \phi(\xi, \psi) \) be a function which we write in terms of these coordinates, A necessary condition for analyticity of \( \phi(\xi, \psi) \) is that the complex derivatives in the coordinate directions exist and are equal. Equating the derivatives in the \( \xi \) and \( \psi \) directions gives us the **Cauchy-Riemann equations**.

\[
\left( \frac{\partial \zeta}{\partial \xi} \right)^{-1} \frac{\partial \phi}{\partial \xi} = \left( \frac{\partial \zeta}{\partial \psi} \right)^{-1} \frac{\partial \phi}{\partial \psi}
\]

We could separate this into two equations by equating the real and imaginary parts or the modulus and argument.
Result 8.2.1 A necessary condition for analyticity of $\phi(\xi, \psi)$, where $z = \zeta(\xi, \psi)$, at $z = z_0$ is that the Cauchy-Riemann equations are satisfied in a neighborhood of $z = z_0$.

\[
\left(\frac{\partial \zeta}{\partial \xi}\right)^{-1} \frac{\partial \phi}{\partial \xi} = \left(\frac{\partial \zeta}{\partial \psi}\right)^{-1} \frac{\partial \phi}{\partial \psi}.
\]

(We could equate the real and imaginary parts or the modulus and argument of this to obtain two equations.) A sufficient condition for analyticity of $f(z)$ is that the Cauchy-Riemann equations hold and the first partial derivatives of $\phi$ exist and are continuous in a neighborhood of $z = z_0$.

Below are the Cauchy-Riemann equations for various forms of $f(z)$.

\[
\begin{align*}
 f(z) &= \phi(x, y), & \phi_x &= -i \phi_y, \\
 f(z) &= u(x, y) + iv(x, y), & u_x &= v_y, & u_y &= -v_x, \\
 f(z) &= \phi(r, \theta), & \phi_r &= -\frac{i}{r} \phi_\theta, \\
 f(z) &= u(r, \theta) + iv(r, \theta), & u_r &= \frac{1}{r} v_\theta, & u_\theta &= -r v_r, \\
 f(z) &= R(r, \theta) e^{i \Theta(r, \theta)}, & R_r &= \frac{R}{r} \Theta_\theta, & \frac{1}{r} R_\theta &= -R \Theta_r, \\
 f(z) &= R(x, y) e^{i \Theta(x, y)}, & R_x &= R \Theta_y, & R_y &= -R \Theta_x.
\end{align*}
\]

Example 8.2.2 Consider the Cauchy-Riemann equations for $f(z) = u(r, \theta) + iv(r, \theta)$. From Exercise 8.3 we know that the complex derivative in the polar coordinate directions is

\[
\frac{d}{dz} = e^{-i\theta} \frac{\partial}{\partial r} = -\frac{i}{r} e^{-i\theta} \frac{\partial}{\partial \theta}.
\]
From Result 8.2.1 we have the equation,

\[ e^{-i\theta} \frac{\partial}{\partial r} [u + iv] = -\frac{i}{r} e^{-i\theta} \frac{\partial}{\partial \theta} [u + iv]. \]

We multiply by \( e^{i\theta} \) and equate the real and imaginary components to obtain the Cauchy-Riemann equations.

\[ u_r = \frac{1}{r} v_\theta, \quad u_\theta = -rv_r \]

**Example 8.2.3** Consider the exponential function.

\[ e^z = \phi(x, y) = e^x(\cos y + i \sin(y)) \]

We use the Cauchy-Riemann equations to show that the function is entire.

\[ \phi_x = -i\phi_y \]

\[ e^x(\cos y + i \sin(y)) = -i e^x(- \sin y + i \cos(y)) \]

\[ e^x(\cos y + i \sin(y)) = e^x(\cos y + i \sin(y)) \]

Since the function satisfies the Cauchy-Riemann equations and the first partial derivatives are continuous everywhere in the finite complex plane, the exponential function is entire.

Now we find the value of the complex derivative.

\[ \frac{d}{dz} e^z = \frac{\partial \phi}{\partial x} = e^x(\cos y + i \sin(y)) = e^z \]

The differentiability of the exponential function implies the differentiability of the trigonometric functions, as they can be written in terms of the exponential.

In Exercise 8.13 you can show that the logarithm \( \log z \) is differentiable for \( z \neq 0 \). This implies the differentiability of \( z^\alpha \) and the inverse trigonometric functions as they can be written in terms of the logarithm.
Example 8.2.4  We compute the derivative of \( z^{z} \).

\[
\frac{d}{dz} (z^{z}) = \frac{d}{dz} e^{z \log z} \\
= (1 + \log z) e^{z \log z} \\
= (1 + \log z) z^{z} \\
= z^{z} + z^{z} \log z
\]

8.3 Harmonic Functions

A function \( u \) is harmonic if its second partial derivatives exist, are continuous and satisfy Laplace’s equation \( \Delta u = 0 \). (In Cartesian coordinates the Laplacian is \( \Delta u \equiv u_{xx} + u_{yy} \).) If \( f(z) = u + iv \) is an analytic function then \( u \) and \( v \) are harmonic functions. To see why this is so, we start with the Cauchy-Riemann equations.

\[
u_x = v_y, \quad u_y = -v_x
\]

We differentiate the first equation with respect to \( x \) and the second with respect to \( y \). (We assume that \( u \) and \( v \) are twice continuously differentiable. We will see later that they are infinitely differentiable.)

\[
u_{xx} = v_{xy}, \quad u_{yy} = -v_{yx}
\]

Thus we see that \( u \) is harmonic.

\[
\Delta u \equiv u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0
\]

One can use the same method to show that \( \Delta v = 0 \).

If \( u \) is harmonic on some simply-connected domain, then there exists a harmonic function \( v \) such that \( f(z) = u + iv \) is analytic in the domain. \( v \) is called the harmonic conjugate of \( u \). The harmonic conjugate is unique up to an additive

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\(^2\) The capital Greek letter \( \Delta \) is used to denote the Laplacian, like \( \Delta u(x, y) \), and differentials, like \( \Delta x \).
constant. To demonstrate this, let \( w \) be another harmonic conjugate of \( u \). Both the pair \( u \) and \( v \) and the pair \( u \) and \( w \) satisfy the Cauchy-Riemann equations.

\[
\begin{align*}
    u_x &= v_y, & u_y &= -v_x, & u_x &= w_y, & u_y &= -w_x
\end{align*}
\]

We take the difference of these equations.

\[
\begin{align*}
    v_x - w_x &= 0, & v_y - w_y &= 0
\end{align*}
\]

On a simply connected domain, the difference between \( v \) and \( w \) is thus a constant.

To prove the existence of the harmonic conjugate, we first write \( v \) as an integral.

\[
v(x, y) = v(x_0, y_0) + \int_{(x_0,y_0)}^{(x,y)} v_x \, dx + v_y \, dy
\]

On a simply connected domain, the integral is path independent and defines a unique \( v \) in terms of \( v_x \) and \( v_y \). We use the Cauchy-Riemann equations to write \( v \) in terms of \( u_x \) and \( u_y \).

\[
v(x, y) = v(x_0, y_0) + \int_{(x_0,y_0)}^{(x,y)} -u_y \, dx + u_x \, dy
\]

Changing the starting point \((x_0, y_0)\) changes \( v \) by an additive constant. The harmonic conjugate of \( u \) to within an additive constant is

\[
v(x, y) = \int -u_y \, dx + u_x \, dy.
\]

This proves the existence\(^3\) of the harmonic conjugate. This is not the formula one would use to construct the harmonic conjugate of a \( u \). One accomplishes this by solving the Cauchy-Riemann equations.

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\(^3\) A mathematician returns to his office to find that a cigarette tossed in the trash has started a small fire. Being calm and a quick thinker he notes that there is a fire extinguisher by the window. He then closes the door and walks away because “the solution exists.”
If \( f(z) = u + iv \) is an analytic function then \( u \) and \( v \) are harmonic functions. That is, the Laplacians of \( u \) and \( v \) vanish \( \Delta u = \Delta v = 0 \). The Laplacian in Cartesian and polar coordinates is

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]

Given a harmonic function \( u \) in a simply connected domain, there exists a harmonic function \( v \), (unique up to an additive constant), such that \( f(z) = u + iv \) is analytic in the domain. One can construct \( v \) by solving the Cauchy-Riemann equations.

**Example 8.3.1** Is \( x^2 \) the real part of an analytic function?

The Laplacian of \( x^2 \) is

\[
\Delta[x^2] = 2 + 0
\]

\( x^2 \) is not harmonic and thus is not the real part of an analytic function.

**Example 8.3.2** Show that \( u = e^{-x}(x \sin y - y \cos y) \) is harmonic.

\[
\frac{\partial u}{\partial x} = e^{-x} \sin y - e^x (x \sin y - y \cos y) = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y
\]

\[
\frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y - e^{-x} \sin y + x e^{-x} \sin y - y e^{-x} \cos y
\]

\[
= -2 e^{-x} \sin y + x e^{-x} \sin y - y e^{-x} \cos y
\]

\[
\frac{\partial u}{\partial y} = e^{-x} (x \cos y - \cos y + y \sin y)
\]
\[
\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x \sin y + \sin y + y \cos y + \sin y)
\]
\[
= -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y
\]

Thus we see that \(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0\) and \(u\) is harmonic.

**Example 8.3.3** Consider \(u = \cos x \cosh y\). This function is harmonic.

\[
u_{xx} + u_{yy} = -\cos x \cosh y + \cos x \cosh y = 0
\]

Thus it is the real part of an analytic function, \(f(z)\). We find the harmonic conjugate, \(v\), with the Cauchy-Riemann equations. We integrate the first Cauchy-Riemann equation.

\[
\begin{align*}
v_y &= u_x = -\sin x \cosh y \\
v_x &= -\sin x \sinh y + a(x)
\end{align*}
\]

Here \(a(x)\) is a constant of integration. We substitute this into the second Cauchy-Riemann equation to determine \(a(x)\).

\[
\begin{align*}
v_x &= -u_y \\
-\cos x \sinh y + a'(x) &= -\cos x \sinh y \\
a'(x) &= 0 \\
a(x) &= c
\end{align*}
\]

Here \(c\) is a real constant. Thus the harmonic conjugate is

\[
v = -\sin x \sinh y + c.
\]

The analytic function is

\[
f(z) = \cos x \cosh y - i \sin x \sinh y + ic
\]

We recognize this as

\[
f(z) = \cos z + ic.
\]
Example 8.3.4 Here we consider an example that demonstrates the need for a simply connected domain. Consider $u = \log r$ in the multiply connected domain, $r > 0$. $u$ is harmonic.

$$\Delta \log r = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \log r \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \log r = 0$$

We solve the Cauchy-Riemann equations to try to find the harmonic conjugate.

$$u_r = \frac{1}{r} v_\theta, \quad u_\theta = -rv_r$$

$$v_r = 0, \quad v_\theta = 1$$

$$v = \theta + c$$

We are able to solve for $v$, but it is multi-valued. Any single-valued branch of $\theta$ that we choose will not be continuous on the domain. Thus there is no harmonic conjugate of $u = \log r$ for the domain $r > 0$.

If we had instead considered the simply-connected domain $r > 0, |\arg(z)| < \pi$ then the harmonic conjugate would be $v = \text{Arg}(z) + c$. The corresponding analytic function is $f(z) = \log z + ic$.

Example 8.3.5 Consider $u = x^3 - 3xy^2 + x$. This function is harmonic.

$$u_{xx} + u_{yy} = 6x - 6x = 0$$

Thus it is the real part of an analytic function, $f(z)$. We find the harmonic conjugate, $v$, with the Cauchy-Riemann equations. We integrate the first Cauchy-Riemann equation.

$$v_y = u_x = 3x^2 - 3y^2 + 1$$

$$v = 3x^2y - y^3 + y + a(x)$$

Here $a(x)$ is a constant of integration. We substitute this into the second Cauchy-Riemann equation to determine $a(x)$.

$$v_x = -u_y$$

$$6xy + a'(x) = 6xy$$

$$a'(x) = 0$$

$$a(x) = c$$

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Here $c$ is a real constant. The harmonic conjugate is

$$v = 3x^2y - y^3 + y + c.$$ 

The analytic function is

\[
\begin{align*}
  f(z) &= x^3 - 3xy^2 + x + i(3x^2y - y^3 + y) + ic \\
  f(z) &= x^3 + 3x^2y - 3xy^2 - iy^3 + x + iy + ic \\
  f(z) &= z^3 + z + ic
\end{align*}
\]

### 8.4 Singularities

Any point at which a function is not analytic is called a *singularity*. In this section we will classify the different flavors of singularities.

**Result 8.4.1 Singularities.** If a function is not analytic at a point, then that point is a *singular point* or a *singularity* of the function.

### 8.4.1 Categorization of Singularities

#### Branch Points.

If $f(z)$ has a branch point at $z_0$, then we cannot define a branch of $f(z)$ that is continuous in a neighborhood of $z_0$. Continuity is necessary for analyticity. Thus all branch points are singularities. Since function are discontinuous across branch cuts, all points on a branch cut are singularities.

**Example 8.4.1** Consider $f(z) = z^{3/2}$. The origin and infinity are branch points and are thus singularities of $f(z)$. We choose the branch $g(z) = \sqrt{z^3}$. All the points on the negative real axis, including the origin, are singularities of $g(z)$.

#### Removable Singularities.
Example 8.4.2 Consider

\[ f(z) = \frac{\sin z}{z}. \]

This function is undefined at \( z = 0 \) because \( f(0) \) is the indeterminate form \( 0/0 \). \( f(z) \) is analytic everywhere in the finite complex plane except \( z = 0 \). Note that the limit as \( z \to 0 \) of \( f(z) \) exists.

\[
\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1
\]

If we were to fill in the hole in the definition of \( f(z) \), we could make it differentiable at \( z = 0 \). Consider the function

\[
g(z) = \begin{cases} 
\frac{\sin z}{z} & z \neq 0, \\
1 & z = 0.
\end{cases}
\]

We calculate the derivative at \( z = 0 \) to verify that \( g(z) \) is analytic there.

\[
f'(0) = \lim_{z \to 0} \frac{f(0) - f(z)}{z} \\
= \lim_{z \to 0} \frac{1 - \sin(z)/z}{z} \\
= \lim_{z \to 0} \frac{z - \sin(z)}{z^2} \\
= \lim_{z \to 0} \frac{1 - \cos(z)}{2z} \\
= \lim_{z \to 0} \frac{\sin(z)}{2} \\
= 0
\]

We call the point at \( z = 0 \) a removable singularity of \( \sin(z)/z \) because we can remove the singularity by defining the value of the function to be its limiting value there.
Consider a function \( f(z) \) that is analytic in a deleted neighborhood of \( z = z_0 \). If \( f(z) \) is not analytic at \( z_0 \), but \( \lim_{z \to z_0} f(z) \) exists, then the function has a removable singularity at \( z_0 \). The function
\[
g(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{z \to z_0} f(z) & z = z_0 \end{cases}
\]
is analytic in a neighborhood of \( z = z_0 \). We show this by calculating \( g'(z_0) \).
\[
g'(z_0) = \lim_{z \to z_0} \frac{g(z_0) - g(z)}{z_0 - z} = \lim_{z \to z_0} \frac{-g'(z)}{-1} = \lim_{z \to z_0} f'(z)
\]
This limit exists because \( f(z) \) is analytic in a deleted neighborhood of \( z = z_0 \).

**Poles.** If a function \( f(z) \) behaves like \( c/ (z - z_0)^n \) near \( z = z_0 \) then the function has an \( n \)th order pole at that point. More mathematically we say
\[
\lim_{z \to z_0} (z - z_0)^n f(z) = c \neq 0.
\]
We require the constant \( c \) to be nonzero so we know that it is not a pole of lower order. We can denote a removable singularity as a pole of order zero.

Another way to say that a function has an \( n \)th order pole is that \( f(z) \) is not analytic at \( z = z_0 \), but \( (z - z_0)^n f(z) \) is either analytic or has a removable singularity at that point.

**Example 8.4.3** \( 1/ \sin (z^2) \) has a second order pole at \( z = 0 \) and first order poles at \( z = (n\pi)^{1/2}, n \in \mathbb{Z}^+ \).

\[
\lim_{z \to 0} \frac{z^2}{\sin (z^2)} = \lim_{z \to 0} \frac{2z}{2z \cos (z^2)} = \lim_{z \to 0} \frac{2}{2 \cos (z^2) - 4z^2 \sin (z^2)} = 1
\]
\[
\lim_{z \to (n\pi)^{1/2}} \frac{z - (n\pi)^{1/2}}{\sin (z^2)} = \lim_{z \to (n\pi)^{1/2}} \frac{1}{2z \cos (z^2)} = \frac{1}{2(n\pi)^{1/2}(-1)^n}
\]

**Example 8.4.4** \(e^{1/z}\) is singular at \(z = 0\). The function is not analytic as \(\lim_{z \to 0} e^{1/z}\) does not exist. We check if the function has a pole of order \(n\) at \(z = 0\).

\[
\lim_{z \to 0} z^n e^{1/z} = \lim_{\zeta \to \infty} \frac{e^\zeta}{\zeta^n} = \lim_{\zeta \to \infty} \frac{e^\zeta}{n!}
\]

Since the limit does not exist for any value of \(n\), the singularity is not a pole. We could say that \(e^{1/z}\) is more singular than any power of \(1/z\).

**Essential Singularities.** If a function \(f(z)\) is singular at \(z = z_0\), but the singularity is not a branch point, or a pole, the the point is an essential singularity of the function.

**The point at infinity.** We can consider the point at infinity \(z \to \infty\) by making the change of variables \(z = 1/\zeta\) and considering \(\zeta \to 0\). If \(f(1/\zeta)\) is analytic at \(\zeta = 0\) then \(f(z)\) is analytic at infinity. We have encountered branch points at infinity before (Section 7.9). Assume that \(f(z)\) is not analytic at infinity. If \(\lim_{z \to \infty} f(z)\) exists then \(f(z)\) has a removable singularity at infinity. If \(\lim_{z \to \infty} f(z)/z^n = c \neq 0\) then \(f(z)\) has an \(n^{th}\) order pole at infinity.
Result 8.4.2 Categorization of Singularities. Consider a function $f(z)$ that has a singularity at the point $z = z_0$. Singularities come in four flavors:

**Branch Points.** Branch points of multi-valued functions are singularities.

**Removable Singularities.** If $\lim_{z \to z_0} f(z)$ exists, then $z_0$ is a removable singularity. It is thus named because the singularity could be removed and thus the function made analytic at $z_0$ by redefining the value of $f(z_0)$.

**Poles.** If $\lim_{z \to z_0} (z - z_0)^n f(z) = \text{const} \neq 0$ then $f(z)$ has an $n^{\text{th}}$ order pole at $z_0$.

**Essential Singularities.** Instead of defining what an essential singularity is, we say what it is not. If $z_0$ neither a branch point, a removable singularity nor a pole, it is an essential singularity.

A pole may be called a non-essential singularity. This is because multiplying the function by an integral power of $z - z_0$ will make the function analytic. Then an essential singularity is a point $z_0$ such that there does not exist an $n$ such that $(z - z_0)^n f(z)$ is analytic there.

8.4.2 Isolated and Non-Isolated Singularities

**Result 8.4.3 Isolated and Non-Isolated Singularities.** Suppose $f(z)$ has a singularity at $z_0$. If there exists a deleted neighborhood of $z_0$ containing no singularities then the point is an isolated singularity. Otherwise it is a non-isolated singularity.
If you don’t like the abstract notion of a deleted neighborhood, you can work with a deleted circular neighborhood. However, this will require the introduction of more math symbols and a Greek letter. \( z = z_0 \) is an isolated singularity if there exists a \( \delta > 0 \) such that there are no singularities in \( 0 < |z - z_0| < \delta \).

**Example 8.4.5** We classify the singularities of \( f(z) = z / \sin z \).

\( z \) has a simple zero at \( z = 0 \). \( \sin z \) has simple zeros at \( z = n\pi \). Thus \( f(z) \) has a removable singularity at \( z = 0 \) and has first order poles at \( z = n\pi \) for \( n \in \mathbb{Z}^\pm \). We can corroborate this by taking limits.

\[
\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\sin z} = \lim_{z \to 0} \frac{1}{\cos z} = 1
\]

\[
\lim_{z \to n\pi} (z - n\pi)f(z) = \lim_{z \to n\pi} \frac{(z - n\pi)z}{\sin z} = \lim_{z \to n\pi} \frac{2z - n\pi}{\cos z} = \frac{(-1)^n}{n\pi} \neq 0
\]

Now to examine the behavior at infinity. There is no neighborhood of infinity that does not contain first order poles of \( f(z) \). (Another way of saying this is that there does not exist an \( R \) such that there are no singularities in \( R < |z| < \infty \).) Thus \( z = \infty \) is a non-isolated singularity.

We could also determine this by setting \( \zeta = 1/z \) and examining the point \( \zeta = 0 \). \( f(1/\zeta) \) has first order poles at \( \zeta = 1/(n\pi) \) for \( n \in \mathbb{Z} \setminus \{0\} \). These first order poles come arbitrarily close to the point \( \zeta = 0 \) There is no deleted neighborhood of \( \zeta = 0 \) which does not contain singularities. Thus \( \zeta = 0 \), and hence \( z = \infty \) is a non-isolated singularity.

The point at infinity is an essential singularity. It is certainly not a branch point or a removable singularity. It is not a pole, because there is no \( n \) such that \( \lim_{z \to \infty} z^{-n} f(z) = \text{const} \neq 0 \). \( z^{-n} f(z) \) has first order poles in any neighborhood of infinity, so this limit does not exist.
8.5 Application: Potential Flow

Example 8.5.1 We consider 2 dimensional uniform flow in a given direction. The flow corresponds to the complex potential
\[
\Phi(z) = v_0 e^{-i\theta_0} z,
\]
where \( v_0 \) is the fluid speed and \( \theta_0 \) is the direction. We find the velocity potential \( \phi \) and stream function \( \psi \).

\[
\Phi(z) = \phi + i\psi
\]
\[
\phi = v_0 (\cos(\theta_0) x + \sin(\theta_0) y), \quad \psi = v_0 (-\sin(\theta_0) x + \cos(\theta_0) y)
\]

These are plotted in Figure 8.1 for \( \theta_0 = \pi/6 \).

![Figure 8.1: The velocity potential \( \phi \) and stream function \( \psi \) for \( \Phi(z) = v_0 e^{-i\theta_0} z \).](image)

Next we find the stream lines, \( \psi = c \).

\[
v_0 (-\sin(\theta_0) x + \cos(\theta_0) y) = c
\]
\[
y = \frac{c}{v_0 \cos(\theta_0)} + \tan(\theta_0) x
\]
Figure 8.2: Streamlines for $\psi = v_0(\sin(\theta_0)x + \cos(\theta_0)y)$.

*Figure 8.2* shows how the streamlines go straight along the $\theta_0$ direction. Next we find the velocity field.

\[
\mathbf{v} = \nabla \phi \\
\mathbf{v} = \phi_x \hat{x} + \phi_y \hat{y} \\
\mathbf{v} = v_0 \cos(\theta_0) \hat{x} + v_0 \sin(\theta_0) \hat{y}
\]

The velocity field is shown in *Figure 8.3*.

**Example 8.5.2** Steady, incompressible, inviscid, irrotational flow is governed by the Laplace equation. We consider flow around an infinite cylinder of radius $a$. Because the flow does not vary along the axis of the cylinder, this is a two-dimensional problem. The flow corresponds to the complex potential

\[
\Phi(z) = v_0 \left( z + \frac{a^2}{z} \right).
\]
We find the velocity potential $\phi$ and stream function $\psi$.

$$\Phi(z) = \phi + \psi$$

$$\phi = v_0 \left( r + \frac{a^2}{r} \right) \cos \theta, \quad \psi = v_0 \left( r - \frac{a^2}{r} \right) \sin \theta$$

These are plotted in Figure 8.4.
Figure 8.4: The velocity potential $\phi$ and stream function $\psi$ for $\Phi(z) = v_0 \left(z + \frac{a^2}{z}\right)$.

Next we find the stream lines, $\psi = c$.

$$v_0 \left(r - \frac{a^2}{r}\right) \sin \theta = c$$

$$r = \frac{c \pm \sqrt{c^2 + 4v_0 \sin^2 \theta}}{2v_0 \sin \theta}$$

Figure 8.5 shows how the streamlines go around the cylinder. Next we find the velocity field.
Figure 8.5: Streamlines for $\psi = v_0 \left( r - \frac{a^2}{r} \right) \sin \theta$.

$$v = \nabla \phi$$

$$v = \phi_r \hat{r} + \frac{\phi_\theta}{r} \hat{\theta}$$

$$v = v_0 \left( 1 - \frac{a^2}{r^2} \right) \cos \theta \hat{r} - v_0 \left( 1 + \frac{a^2}{r^2} \right) \sin \theta \hat{\theta}$$

The velocity field is shown in Figure 8.6.