Chapter 11

Cauchy’s Integral Formula

If I were founding a university I would begin with a smoking room; next a dormitory; and then a decent reading room and a library. After that, if I still had more money that I couldn’t use, I would hire a professor and get some text books.

- Stephen Leacock
11.1 Cauchy’s Integral Formula

Result 11.1.1 Cauchy’s Integral Formula. If \( f(\zeta) \) is analytic in a compact, closed, connected domain \( D \) and \( z \) is a point in the interior of \( D \) then

\[
f(z) = \frac{1}{i2\pi} \oint_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{i2\pi} \sum_k \oint_{C_k} \frac{f(\zeta)}{\zeta - z} \, d\zeta. \tag{11.1}
\]

Here the set of contours \( \{C_k\} \) make up the positively oriented boundary \( \partial D \) of the domain \( D \). More generally, we have

\[
f^{(n)}(z) = \frac{n!}{i2\pi} \oint_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta = \frac{n!}{i2\pi} \sum_k \oint_{C_k} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \tag{11.2}
\]

Cauchy’s Formula shows that the value of \( f(z) \) and all its derivatives in a domain are determined by the value of \( f(z) \) on the boundary of the domain. Consider the first formula of the result, Equation 11.1. We deform the contour to a circle of radius \( \delta \) about the point \( \zeta = z \).

\[
\oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = \oint_{C_\delta} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]

\[
= \oint_{C_\delta} f(z) \, d\zeta + \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta
\]

We use the result of Example 10.8.1 to evaluate the first integral.

\[
\oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = i2\pi f(z) + \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta
\]
The remaining integral along $C_\delta$ vanishes as $\delta \to 0$ because $f(\zeta)$ is continuous. We demonstrate this with the maximum modulus integral bound. The length of the path of integration is $2\pi \delta$.

$$
\lim_{\delta \to 0} \left| \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta \right| \leq \lim_{\delta \to 0} \left( (2\pi \delta)^{1/\delta} \max_{|\zeta - z| = \delta} |f(\zeta) - f(z)| \right)
$$

$$
\leq \lim_{\delta \to 0} \left( 2\pi \max_{|\zeta - z| = \delta} |f(\zeta) - f(z)| \right)
$$

$$
= 0
$$

This gives us the desired result.

$$
f(z) = \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta
$$

We derive the second formula, Equation 11.2, from the first by differentiating with respect to $z$. Note that the integral converges uniformly for $z$ in any closed subset of the interior of $C$. Thus we can differentiate with respect to $z$ and interchange the order of differentiation and integration.

$$
f^{(n)}(z) = \frac{1}{i2\pi} \frac{d^n}{dz^n} \oint_C \frac{f(\zeta)}{\zeta - z} \, d\zeta
$$

$$
= \frac{1}{i2\pi} \oint_C \frac{d^n}{dz^n} \frac{f(\zeta)}{\zeta - z} \, d\zeta
$$

$$
= \frac{n!}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta
$$

**Example 11.1.1** Consider the following integrals where $C$ is the positive contour on the unit circle. For the third integral, the point $z = -1$ is removed from the contour.

1. $\oint_C \sin(\cos(z^5)) \, dz$

2. $\oint_C \frac{1}{(z - 3)(3z - 1)} \, dz$
3. \[
\int_C \sqrt{z} \, dz
\]

1. Since \( \sin (\cos (z^5)) \) is an analytic function inside the unit circle,
\[
\oint_C \sin (\cos (z^5)) \, dz = 0
\]

2. \( \frac{1}{(z-3)(3z-1)} \) has singularities at \( z = 3 \) and \( z = 1/3 \). Since \( z = 3 \) is outside the contour, only the singularity at \( z = 1/3 \) will contribute to the value of the integral. We will evaluate this integral using the Cauchy integral formula.
\[
\oint_C \frac{1}{(z-3)(3z-1)} \, dz = i2\pi \left( \frac{1}{1/3-3} \right) = -\frac{i\pi}{4}
\]

3. Since the curve is not closed, we cannot apply the Cauchy integral formula. Note that \( \sqrt{z} \) is single-valued and analytic in the complex plane with a branch cut on the negative real axis. Thus we use the Fundamental Theorem of Calculus.
\[
\int_C \sqrt{z} \, dz = \left[ \frac{2}{3} \sqrt{z^3} \right]_{e^{i\pi}}^{e^{-i\pi}}
\]
\[
= \frac{2}{3} \left( e^{i3\pi/2} - e^{-i3\pi/2} \right)
\]
\[
= \frac{2}{3} (-i - i)
\]
\[
= -\frac{4i}{3}
\]

Cauchy’s Inequality. Suppose the \( f(\zeta) \) is analytic in the closed disk \( |\zeta - z| \leq r \). By Cauchy’s integral formula,
\[
f^{(n)}(z) = \frac{n!}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta,
\]
where $C$ is the circle of radius $r$ centered about the point $z$. We use this to obtain an upper bound on the modulus of $f^{(n)}(z)$.

$$|f^{(n)}(z)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \right|$$

$$\leq \frac{n!}{2\pi r} \max_{|\zeta - z| = r} \left| \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right|$$

$$= \frac{n!}{r^n} \max_{|\zeta - z| = r} |f(\zeta)|$$

**Result 11.1.2 Cauchy’s Inequality.** If $f(\zeta)$ is analytic in $|\zeta - z| \leq r$ then

$$|f^{(n)}(z)| \leq \frac{n!M}{r^n}$$

where $|f(\zeta)| \leq M$ for all $|\zeta - z| = r$.

**Liouville’s Theorem.** Consider a function $f(z)$ that is analytic and bounded, ($|f(z)| \leq M$), in the complex plane. From Cauchy’s inequality,

$$|f'(z)| \leq \frac{M}{r}$$

for any positive $r$. By taking $r \to \infty$, we see that $f'(z)$ is identically zero for all $z$. Thus $f(z)$ is a constant.

**Result 11.1.3 Liouville’s Theorem.** If $f(z)$ is analytic and $|f(z)|$ is bounded in the complex plane then $f(z)$ is a constant.

**The Fundamental Theorem of Algebra.** We will prove that every polynomial of degree $n \geq 1$ has exactly $n$ roots, counting multiplicities. First we demonstrate that each such polynomial has at least one root. Suppose that an
\(n\)th degree polynomial \(p(z)\) has no roots. Let the lower bound on the modulus of \(p(z)\) be \(0 < m \leq |p(z)|\). The function \(f(z) = 1/p(z)\) is analytic, \((f'(z) = p'(z)/p^2(z))\), and bounded, \(|f(z)| \leq 1/m\), in the extended complex plane. Using Liouville’s theorem we conclude that \(f(z)\) and hence \(p(z)\) are constants, which yields a contradiction. Therefore every such polynomial \(p(z)\) must have at least one root.

Now we show that we can factor the root out of the polynomial. Let

\[
p(z) = \sum_{k=0}^{n} p_k z^k.
\]

We note that

\[
(z^n - c^n) = (z - c) \sum_{k=0}^{n-1} c^{n-1-k} z^k.
\]

Suppose that the \(n\)th degree polynomial \(p(z)\) has a root at \(z = c\).

\[
p(z) = p(z) - p(c)
\]

\[
= \sum_{k=0}^{n} p_k z^k - \sum_{k=0}^{n} p_k c^k
\]

\[
= \sum_{k=0}^{n} p_k (z^k - c^k)
\]

\[
= \sum_{k=0}^{n} p_k (z - c) \sum_{j=0}^{k-1} c^{k-1-j} z^j
\]

\[
= (z - c) q(z)
\]

Here \(q(z)\) is a polynomial of degree \(n - 1\). By induction, we see that \(p(z)\) has exactly \(n\) roots.

**Result 11.1.4 Fundamental Theorem of Algebra.** Every polynomial of degree \(n \geq 1\) has exactly \(n\) roots, counting multiplicities.
Gauss’ Mean Value Theorem. Let $f(ζ)$ be analytic in $|ζ − z| ≤ r$. By Cauchy’s integral formula,

$$f(z) = \frac{1}{i2\pi} \oint_C \frac{f(ζ)}{ζ − z} \, dζ,$$

where $C$ is the circle $|ζ − z| = r$. We parameterize the contour with $ζ = z + re^{iθ}$.

$$f(z) = \frac{1}{i2\pi} \int_0^{2\pi} \frac{f(z + re^{iθ})}{re^{iθ}} i r e^{iθ} \, dθ$$

Writing this in the form,

$$f(z) = \frac{1}{2\pi r} \int_0^{2\pi} f(z + re^{iθ}) r \, dθ,$$

we see that $f(z)$ is the average value of $f(ζ)$ on the circle of radius $r$ about the point $z$.

**Result 11.1.5 Gauss’ Average Value Theorem.** If $f(ζ)$ is analytic in $|ζ − z| ≤ r$ then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{iθ}) \, dθ.$$

That is, $f(z)$ is equal to its average value on a circle of radius $r$ about the point $z$.

Extremum Modulus Theorem. Let $f(z)$ be analytic in closed, connected domain, $D$. The extreme values of the modulus of the function must occur on the boundary. If $|f(z)|$ has an interior extrema, then the function is a constant. We will show this with proof by contradiction. Assume that $|f(z)|$ has an interior maxima at the point $z = c$. This means that there exists an neighborhood of the point $z = c$ for which $|f(z)| ≤ |f(c)|$. Choose an $ε$ so that the set $|z − c| ≤ ε$ lies inside this neighborhood. First we use Gauss’ mean value theorem.

$$f(c) = \frac{1}{2\pi} \int_0^{2\pi} f(c + εe^{iθ}) \, dθ$$
We get an upper bound on $|f(c)|$ with the maximum modulus integral bound.

$$|f(c)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(c + e^{i\theta})| \, d\theta$$

Since $z = c$ is a maxima of $|f(z)|$ we can get a lower bound on $|f(c)|$.

$$|f(c)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(c + e^{i\theta})| \, d\theta$$

If $|f(z)| < |f(c)|$ for any point on $|z - c| = \epsilon$, then the continuity of $f(z)$ implies that $|f(z)| < |f(c)|$ in a neighborhood of that point which would make the value of the integral of $|f(z)|$ strictly less than $|f(c)|$. Thus we conclude that $|f(z)| = |f(c)|$ for all $|z - c| = \epsilon$. Since we can repeat the above procedure for any circle of radius smaller than $\epsilon$, $|f(z)| = |f(c)|$ for all $|z - c| \leq \epsilon$, i.e. all the points in the disk of radius $\epsilon$ about $z = c$ are also maxima. By recursively repeating this procedure points in this disk, we see that $|f(z)| = |f(c)|$ for all $z \in D$. This implies that $f(z)$ is a constant in the domain. By reversing the inequalities in the above method we see that the minimum modulus of $f(z)$ must also occur on the boundary.

**Result 11.1.6 Extremum Modulus Theorem.** Let $f(z)$ be analytic in a closed, connected domain, $D$. The extreme values of the modulus of the function must occur on the boundary. If $|f(z)|$ has an interior extrema, then the function is a constant.
11.2 The Argument Theorem

**Result 11.2.1 The Argument Theorem.** Let $f(z)$ be analytic inside and on $C$ except for isolated poles inside the contour. Let $f(z)$ be nonzero on $C$.

$$\frac{1}{i2\pi} \int_C \frac{f'(z)}{f(z)} \, dz = N - P$$

Here $N$ is the number of zeros and $P$ the number of poles, counting multiplicities, of $f(z)$ inside $C$.

First we will simplify the problem and consider a function $f(z)$ that has one zero or one pole. Let $f(z)$ be analytic and nonzero inside and on $A$ except for a zero of order $n$ at $z = a$. Then we can write $f(z) = (z - a)^n g(z)$ where $g(z)$ is analytic and nonzero inside and on $A$. The integral of $\frac{f'(z)}{f(z)}$ along $A$ is

$$\frac{1}{i2\pi} \int_A \frac{f'(z)}{f(z)} \, dz = \frac{1}{i2\pi} \int_A \frac{d}{dz} \left( \log(f(z)) \right) \, dz$$

$$= \frac{1}{i2\pi} \int_A \frac{d}{dz} \left( \log((z - a)^n) + \log(g(z)) \right) \, dz$$

$$= \frac{1}{i2\pi} \int_A \frac{d}{dz} \left( \log((z - a)^n) \right) \, dz$$

$$= \frac{1}{i2\pi} \int_A \frac{n}{z - a} \, dz$$

$$= n$$
Now let \( f(z) \) be analytic and nonzero inside and on \( B \) except for a pole of order \( p \) at \( z = b \). Then we can write \( f(z) = \frac{g(z)}{(z-b)^p} \) where \( g(z) \) is analytic and nonzero inside and on \( B \). The integral of \( \frac{f'(z)}{f(z)} \) along \( B \) is

\[
\frac{1}{i2\pi} \int_B \frac{f'(z)}{f(z)} \, dz = \frac{1}{i2\pi} \int_B \frac{d}{dz} (\log(f(z))) \, dz
\]

\[
= \frac{1}{i2\pi} \int_B \frac{d}{dz} (\log((z-b)^{-p}) + \log(g(z))) \, dz
\]

\[
= \frac{1}{i2\pi} \int_B \frac{d}{dz} (\log((z-b)^{-p}) + ) \, dz
\]

\[
= \frac{1}{i2\pi} \int_B \frac{-p}{z-b} \, dz
\]

\[
= -p
\]

Now consider a function \( f(z) \) that is analytic inside an on the contour \( C \) except for isolated poles at the points \( b_1, \ldots, b_p \). Let \( f(z) \) be nonzero except at the isolated points \( a_1, \ldots, a_n \). Let the contours \( A_k, k = 1, \ldots, n \), be simple, positive contours which contain the zero at \( a_k \) but no other poles or zeros of \( f(z) \). Likewise, let the contours \( B_k, k = 1, \ldots, p \) be simple, positive contours which contain the pole at \( b_k \) but no other poles of zeros of \( f(z) \). (See Figure 11.1.) By deforming the contour we obtain

\[
\int_C \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^n \int_{A_j} \frac{f'(z)}{f(z)} \, dz + \sum_{k=1}^p \int_{B_j} \frac{f'(z)}{f(z)} \, dz.
\]

From this we obtain Result 11.2.1.

### 11.3 Rouche’s Theorem

**Result 11.3.1 Rouche’s Theorem.** Let \( f(z) \) and \( g(z) \) be analytic inside and on a simple, closed contour \( C \). If \(|f(z)| > |g(z)|\) on \( C \) then \( f(z) \) and \( f(z) + g(z) \) have the same number of zeros inside \( C \) and no zeros on \( C \).
First note that since $|f(z)| > |g(z)|$ on $C$, $f(z)$ is nonzero on $C$. The inequality implies that $|f(z) + g(z)| > 0$ on $C$ so $f(z) + g(z)$ has no zeros on $C$. We well count the number of zeros of $f(z)$ and $g(z)$ using the Argument Theorem, (Result 11.2.1). The number of zeros $N$ of $f(z)$ inside the contour is

$$N = \frac{1}{i2\pi} \oint_C \frac{f'(z)}{f(z)} \, dz.$$

Now consider the number of zeros $M$ of $f(z) + g(z)$. We introduce the function $h(z) = g(z)/f(z)$.

$$M = \frac{1}{i2\pi} \oint_C \frac{f'(z) + g'(z)}{f(z) + g(z)} \, dz$$

$$= \frac{1}{i2\pi} \oint_C \frac{f'(z) + f'(z)h(z) + f(z)h'(z)}{f(z) + f(z)h(z)} \, dz$$

$$= \frac{1}{i2\pi} \oint_C \frac{f'(z)}{f(z)} \, dz + \frac{1}{i2\pi} \oint_C \frac{h'(z)}{1 + h(z)} \, dz$$

$$= N + \frac{1}{i2\pi} \left[ \log(1 + h(z)) \right]_C$$

$$= N$$
(Note that since $|h(z)| < 1$ on $C$, $\Re(1 + h(z)) > 0$ on $C$ and the value of $\log(1 + h(z))$ does not change in traversing the contour.) This demonstrates that $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $C$ and proves the result.