Chapter 6

Complex Numbers

I’m sorry. You have reached an imaginary number. Please rotate your phone 90 degrees and dial again.

-Message on answering machine of Cathy Vargas.

6.1 Complex Numbers

Shortcomings of real numbers. When you started algebra, you learned that the quadratic equation: $x^2 + 2ax + b = 0$ has either two, one or no solutions. For example:

- $x^2 - 3x + 2 = 0$ has the two solutions $x = 1$ and $x = 2$.
- For $x^2 - 2x + 1 = 0$, $x = 1$ is a solution of multiplicity two.
- $x^2 + 1 = 0$ has no solutions.
This is a little unsatisfactory. We can formally solve the general quadratic equation.

\[ x^2 + 2ax + b = 0 \]

\[ (x + a)^2 = a^2 - b \]

\[ x = -a \pm \sqrt{a^2 - b} \]

However, the solutions are defined only when the discriminant \( a^2 - b \) is non-negative. This is because the square root function \( \sqrt{x} \) is a bijection from \( \mathbb{R}^0 \) to \( \mathbb{R}^0 \). (See Figure 6.1.)

![Figure 6.1: \( y = \sqrt{x} \)](image)

A new mathematical constant. We cannot solve \( x^2 = -1 \) because the square root of \(-1\) is not defined. To overcome this apparent shortcoming of the real number system, we create a new symbolic constant \( \sqrt{-1} \). In performing arithmetic, we will treat \( \sqrt{-1} \) as we would a real constant like \( \pi \) or a formal variable like \( x \), i.e. \( \sqrt{-1} + \sqrt{-1} = 2\sqrt{-1} \). This constant has the property: \( (\sqrt{-1})^2 = -1 \). Now we can express the solutions of \( x^2 = -1 \) as \( x = \sqrt{-1} \) and \( x = -\sqrt{-1} \). These satisfy the equation since \( (\sqrt{-1})^2 = -1 \) and \( (-\sqrt{-1})^2 = (-1)^2 (\sqrt{-1})^2 = -1 \). Note that we can express the square root of any negative real number in terms of \( \sqrt{-1} \): \( \sqrt{-r} = \sqrt{-1} \sqrt{r} \) for \( r \geq 0 \).
Euler’s notation. Euler introduced the notation of using the letter \( i \) to denote \( \sqrt{-1} \). We will use the symbol \( \imath \), an \( i \) without a dot, to denote \( \sqrt{-1} \). This helps us distinguish it from \( i \) used as a variable or index. Let \( a \) and \( b \) be real numbers. The product of a real number and an imaginary number is an imaginary number: \((a)(\imath b) = \imath(ab)\). The product of two imaginary numbers is a real number: \((\imath a)(\imath b) = -ab\). However the sum of a real number and an imaginary number \( a + \imath b \) is neither real nor imaginary. We call numbers of the form \( a + \imath b \) complex numbers.

The quadratic. Now we return to the quadratic with real coefficients, \( x^2 + 2ax + b = 0 \). It has the solutions \( x = -a \pm \sqrt{a^2 - b} \). The solutions are real-valued only if \( a^2 - b \geq 0 \). If not, then we can define solutions as complex numbers. If the discriminant is negative, we write \( x = -a \pm \imath \sqrt{b - a^2} \). Thus every quadratic polynomial with real coefficients has exactly two solutions, counting multiplicities. The fundamental theorem of algebra states that an \( n \)th degree polynomial with complex coefficients has \( n \), not necessarily distinct, complex roots. We will prove this result later using the theory of functions of a complex variable.

Component operations. Consider the complex number \( z = x + \imath y \), \((x, y \in \mathbb{R})\). The real part of \( z \) is \( \Re(z) = x \); the imaginary part of \( z \) is \( \Im(z) = y \). Two complex numbers, \( z = x + \imath y \) and \( \zeta = \xi + \imath \psi \), are equal if and only if \( x = \xi \) and \( y = \psi \). The complex conjugate is \( \bar{z} = x - \imath y \). The notation \( z^* \equiv x - \imath y \) is also used.

A little arithmetic. Consider two complex numbers: \( z = x + \imath y \), \( \zeta = \xi + \imath \psi \). It is easy to express the sum or difference as a complex number.

\[
z + \zeta = (x + \xi) + \imath(y + \psi), \quad z - \zeta = (x - \xi) + \imath(y - \psi)
\]

It is also easy to form the product.

\[
z\zeta = (x + \imath y)(\xi + \imath \psi) = x\xi + \imath x\psi + \imath y\xi + \imath^2 y\psi = (x\xi - y\psi) + \imath(x\psi + y\xi)
\]

---

1 Electrical engineering types prefer to use \( j \) or \( j \) to denote \( \sqrt{-1} \).

2 “Imaginary” is an unfortunate term. Real numbers are artificial; constructs of the mind. Real numbers are no more real than imaginary numbers.

3 Here complex means “composed of two or more parts”, not “hard to separate, analyze, or solve”. Those who disagree have a complex number complex.

4 Conjugate: having features in common but opposite or inverse in some particular.
The quotient is a bit more difficult. (Assume that \( \zeta \) is nonzero.) How do we express \( \frac{z}{\zeta} = \frac{x + iy}{(\xi + i\psi)} \) as the sum of a real number and an imaginary number? The trick is to multiply the numerator and denominator by the complex conjugate of \( \zeta \).

\[
\frac{z}{\zeta} = \frac{x + iy}{\xi + i\psi} = \frac{x\xi - iy\psi - iy\xi - i^2y\psi}{\xi^2 - i\xi\psi + i\psi\xi - i^2\psi^2} = \frac{(x\xi + y\psi) - i(x\psi + y\xi)}{\xi^2 + \psi^2} = \frac{(x\xi + y\psi)}{\xi^2 + \psi^2} - i\frac{x\psi + y\xi}{\xi^2 + \psi^2}
\]

Now we recognize it as a complex number.

**Field properties.** The set of complex numbers \( \mathbb{C} \) form a field. That essentially means that we can do arithmetic with complex numbers. When performing arithmetic, we simply treat \( i \) as a symbolic constant with the property that \( i^2 = -1 \). The field of complex numbers satisfy the following list of properties. Each one is easy to verify; some are proved below. (Let \( z, \zeta, \omega \in \mathbb{C} \).)

1. Closure under addition and multiplication.

\[
z + \zeta = (x + iy) + (\xi + i\psi)
\]
\[
= (x + \xi) + i(y + \psi) \in \mathbb{C}
\]
\[
z\zeta = (x + iy)(\xi + i\psi)
\]
\[
= x\xi + ix\psi + iy\xi + i^2y\psi
\]
\[
= (x\xi - y\psi) + i(x\psi + \xi y) \in \mathbb{C}
\]

2. Commutativity of addition and multiplication. \( z + \zeta = \zeta + z \). \( z\zeta = \zeta z \).

3. Associativity of addition and multiplication. \( (z + \zeta) + \omega = z + (\zeta + \omega) \). \( (z\zeta)\omega = z(\zeta\omega) \).

4. Distributive law. \( z(\zeta + \omega) = z\zeta + z\omega \).

5. Identity with respect to addition and multiplication. Zero is the additive identity element, \( z + 0 = z \); unity is the multiplicative identity element, \( z(1) = z \).

6. Inverse with respect to addition. \( z + (-z) = (x + iy) + (-x - iy) = (x - x) + i(y - y) = 0 \).
7. Inverse with respect to multiplication for nonzero numbers. \( zz^{-1} = 1 \), where

\[
zz^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy} \frac{x - iy}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i.
\]

Properties of the complex conjugate. Using the field properties of complex numbers, we can derive the following properties of the complex conjugate, \( \bar{z} = x - iy \).

1. \( \overline{(z)} = z \),
2. \( z + \zeta = \bar{z} + \bar{\zeta} \),
3. \( z\zeta = \bar{z}\bar{\zeta} \),
4. \( \overline{\left( \frac{z}{\zeta} \right)} = \frac{\overline{z}}{\overline{\zeta}} \).

6.2 The Complex Plane

Complex plane. We can denote a complex number \( z = x + iy \) as an ordered pair of real numbers \((x, y)\). Thus we can represent a complex number as a point in \( \mathbb{R}^2 \) where the first component is the real part and the second component is the imaginary part of \( z \). This is called the complex plane or the Argand diagram. (See Figure 6.2.) A complex number written as \( z = x + iy \) is said to be in Cartesian form, or \( a + ib \) form.

Recall that there are two ways of describing a point in the complex plane: an ordered pair of coordinates \((x, y)\) that give the horizontal and vertical offset from the origin or the distance \( r \) from the origin and the angle \( \theta \) from the positive horizontal axis. The angle \( \theta \) is not unique. It is only determined up to an additive integer multiple of \( 2\pi \).
Modulus. The *magnitude* or *modulus* of a complex number is the distance of the point from the origin. It is defined as \( |z| = |x + iy| = \sqrt{x^2 + y^2} \). Note that \( z \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \). The modulus has the following properties.

1. \( |z \zeta| = |z| |\zeta| \)
2. \( \left| \frac{z}{\zeta} \right| = \frac{|z|}{|\zeta|} \) for \( \zeta \neq 0 \).
3. \( |z + \zeta| \leq |z| + |\zeta| \)
4. \( |z + \zeta| \geq ||z| - |\zeta|| \)

We could prove the first two properties by expanding in \( x + iy \) form, but it would be fairly messy. The proofs will become simple after polar form has been introduced. The second two properties follow from the triangle inequalities in geometry. This will become apparent after the relationship between complex numbers and vectors is introduced. One can show that

\[
|z_1 z_2 \cdots z_n| = |z_1| |z_2| \cdots |z_n|
\]

and

\[
|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|
\]

with proof by induction.
**Argument.** The *argument* of a complex number is the angle that the vector with tail at the origin and head at \( z = x + iy \) makes with the positive \( x \)-axis. The argument is denoted \( \arg(z) \). Note that the argument is defined for all nonzero numbers and is only determined up to an additive integer multiple of \( 2\pi \). That is, the argument of a complex number is the set of values: \( \{ \theta + 2\pi n \mid n \in \mathbb{Z} \} \). The *principal argument* of a complex number is that angle in the set \( \arg(z) \) which lies in the range \( (-\pi, \pi] \). The principal argument is denoted \( \text{Arg}(z) \). We prove the following identities in Exercise 6.10.

\[
\begin{align*}
\arg(z\zeta) &= \arg(z) + \arg(\zeta) \\
\text{Arg}(z\zeta) &\neq \text{Arg}(z) + \text{Arg}(\zeta) \\
\arg(z^2) &= \arg(z) + \arg(z) \neq 2\arg(z)
\end{align*}
\]

**Example 6.2.1** Consider the equation \( |z - 1 - i| = 2 \). The set of points satisfying this equation is a circle of radius 2 and center at \( 1 + i \) in the complex plane. You can see this by noting that \( |z - 1 - i| \) is the distance from the point \( (1, 1) \). (See Figure 6.3.)

![Figure 6.3: Solution of \( |z - 1 - i| = 2 \).](image-url)
Another way to derive this is to substitute \( z = x + iy \) into the equation.

\[
|x + iy - 1 - i| = 2 \\
\sqrt{(x - 1)^2 + (y - 1)^2} = 2 \\
(x - 1)^2 + (y - 1)^2 = 4
\]

This is the analytic geometry equation for a circle of radius 2 centered about \((1, 1)\).

**Example 6.2.2** Consider the curve described by

\( |z| + |z - 2| = 4 \).

Note that \(|z|\) is the distance from the origin in the complex plane and \(|z - 2|\) is the distance from \(z = 2\). The equation is

\[
(distance \ from \ (0, 0)) + (distance \ from \ (2, 0)) = 4.
\]

From geometry, we know that this is an ellipse with foci at \((0, 0)\) and \((2, 0)\), major axis 2, and minor axis \(\sqrt{3}\). (See Figure 6.4.)

We can use the substitution \( z = x + iy \) to get the equation in algebraic form.

\[
|x| + |x - 2| = 4 \\
|x + iy| + |x + iy - 2| = 4 \\
\sqrt{x^2 + y^2} + \sqrt{(x - 2)^2 + y^2} = 4 \\
x^2 + y^2 = 16 - 8\sqrt{(x - 2)^2 + y^2} + x^2 - 4x + 4 + y^2 \\
x - 5 = -2\sqrt{(x - 2)^2 + y^2} \\
x^2 - 10x + 25 = 4x^2 - 16x + 16 + 4y^2 \\
\frac{1}{4}(x - 1)^2 + \frac{1}{3}y^2 = 1
\]

Thus we have the standard form for an equation describing an ellipse.
Figure 6.4: Solution of $|z| + |z - 2| = 4$.

### 6.3 Polar Form

**Polar form.** A complex number written in Cartesian form, $z = x + iy$, can be converted **polar form**, $z = r(\cos \theta + i\sin \theta)$, using trigonometry. Here $r = |z|$ is the modulus and $\theta = \arctan(x, y)$ is the argument of $z$. The argument is the angle between the $x$ axis and the vector with its head at $(x, y)$. (See Figure 6.5.) Note that $\theta$ is not unique. If $z = r(\cos \theta + i\sin \theta)$ then $z = r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$ for any $n \in \mathbb{Z}$.

**The arctangent.** Note that $\arctan(x, y)$ is not the same thing as the old arctangent that you learned about in trigonometry $\arctan(x, y)$ is sensitive to the quadrant of the point $(x, y)$, while $\arctan \left( \frac{y}{x} \right)$ is not. For example,

\[
\arctan(1, 1) = \frac{\pi}{4} + 2n\pi \quad \text{and} \quad \arctan(-1, -1) = \frac{-3\pi}{4} + 2n\pi,
\]
whereas
\[ \arctan \left( \frac{-1}{-1} \right) = \arctan \left( \frac{1}{1} \right) = \arctan(1). \]

**Euler’s formula.** *Euler’s formula,* \( e^{i\theta} = \cos \theta + i \sin \theta, \) \(^5\) allows us to write the polar form more compactly. Expressing the polar form in terms of the exponential function of imaginary argument makes arithmetic with complex numbers much more convenient.

\[ z = r(\cos \theta + i \sin \theta) = r e^{i\theta} \]

The exponential of an imaginary argument has all the nice properties that we know from studying functions of a real variable, like \( e^{a} e^{b} = e^{(a+b)}. \) Later on we will introduce the exponential of a complex number.

Using Euler’s Formula, we can express the cosine and sine in terms of the exponential.

\[
\begin{align*}
\frac{e^{i\theta} + e^{-i\theta}}{2} &= \frac{(\cos(\theta) + i \sin(\theta)) + (\cos(-\theta) + i \sin(-\theta))}{2} = \cos(\theta) \\
\frac{e^{i\theta} - e^{-i\theta}}{i2} &= \frac{(\cos(\theta) + i \sin(\theta)) - (\cos(-\theta) + i \sin(-\theta))}{i2} = \sin(\theta)
\end{align*}
\]

**Arithmetic with complex numbers.** Note that it is convenient to add complex numbers in Cartesian form.

\[ z + \zeta = (x + iy) + (\xi + i\psi) = (x + \xi) + i(y + \psi) \]

\(^5\) See Exercise 6.17 for justification of Euler’s formula.
However, it is difficult to multiply or divide them in Cartesian form.

\[ z\zeta = (x + iy)(\xi + i\psi) = (x\xi - y\psi) + i(x\psi + y\xi) \]
\[ \frac{z}{\zeta} = \frac{x + iy}{\xi + i\psi} = \frac{(x + iy)(\xi - i\psi)}{(\xi + i\psi)(\xi - i\psi)} = \frac{x\xi + y\psi}{\xi^2 + \psi^2} + i\frac{\xi y - x\psi}{\xi^2 + \psi^2} \]

On the other hand, it is difficult to add complex numbers in polar form.

\[ z + \zeta = r e^{i\theta} + \rho e^{i\phi} \]
\[ = r (\cos \theta + i \sin \theta) + \rho (\cos \phi + i \sin \phi) \]
\[ = r \cos \theta + \rho \cos \phi + i (r \sin \theta + \rho \sin \phi) \]
\[ = \sqrt{(r \cos \theta + \rho \cos \phi)^2 + (r \sin \theta + \rho \sin \phi)^2} \]
\[ \times e^{i \arctan(r \cos \theta + \rho \cos \phi, r \sin \theta + \rho \sin \phi)} \]
\[ = \sqrt{r^2 + \rho^2 + 2 \cos (\theta - \phi)} e^{i \arctan(r \cos \theta + \rho \cos \phi, r \sin \theta + \rho \sin \phi)} \]

However, it is convenient to multiply and divide them in polar form.

\[ z\zeta = r e^{i\theta} \rho e^{i\phi} = r\rho e^{i(\theta + \phi)} \]
\[ \frac{z}{\zeta} = \frac{r e^{i\theta}}{\rho e^{i\phi}} = \frac{r}{\rho} e^{i(\theta - \phi)} \]

Keeping this in mind will make working with complex numbers a shade or two less grungy.
**Result 6.3.1** Euler’s formula is

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

We can write the cosine and sine in terms of the exponential.

\[ \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{i2} \]

To change between Cartesian and polar form, use the identities

\[ r e^{i\theta} = r \cos \theta + ir \sin \theta, \]
\[ x + iy = \sqrt{x^2 + y^2} e^{i \arctan(x,y)}. \]

Cartesian form is convenient for addition. Polar form is convenient for multiplication and division.

**Example 6.3.1** We write \(5 + i7\) in polar form.

\[ 5 + i7 = \sqrt{74} e^{i \arctan(5,7)} \]

We write \(2 e^{i\pi/6}\) in Cartesian form.

\[ 2 e^{i\pi/6} = 2 \cos \left(\frac{\pi}{6}\right) + 2i \sin \left(\frac{\pi}{6}\right) \]
\[ = \sqrt{3} + i \]

**Example 6.3.2** We will prove the trigonometric identity

\[ \cos^4 \theta = \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) + \frac{3}{8}. \]
We start by writing the cosine in terms of the exponential.

\[
\cos^4 \theta = \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^4
\]

\[
= \frac{1}{16} \left( e^{4i\theta} + 4 e^{2i\theta} + 6 + 4 e^{-2i\theta} + e^{-4i\theta} \right)
\]

\[
= \frac{1}{8} \left( e^{4i\theta} + e^{-4i\theta} \right) + \frac{1}{2} \left( e^{2i\theta} + e^{-2i\theta} \right) + \frac{3}{8}
\]

\[
= \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) + \frac{3}{8}
\]

By the definition of exponentiation, we have \(e^{n\theta} = (e^{i\theta})^n\). We apply Euler’s formula to obtain a result which is useful in deriving trigonometric identities.

\[
\cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n
\]

**Result 6.3.2 DeMoivre’s Theorem.**

\[
\cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n
\]

---

\(a\)It’s amazing what passes for a theorem these days. I would think that this would be a corollary at most.

**Example 6.3.3** We will express \(\cos(5\theta)\) in terms of \(\cos \theta\) and \(\sin(5\theta)\) in terms of \(\sin \theta\). We start with DeMoivre’s theorem.

\[
e^{i5\theta} = (e^{i\theta})^5
\]
\[
\cos(5\theta) + i\sin(5\theta) = (\cos \theta + i \sin \theta)^5
\]

\[
= \binom{5}{0} \cos^5 \theta + i \binom{5}{1} \cos^4 \theta \sin \theta - \binom{5}{2} \cos^3 \theta \sin^2 \theta - i \binom{5}{3} \cos^2 \theta \sin^3 \theta
+ \binom{5}{4} \cos \theta \sin^4 \theta + i \binom{5}{5} \sin^5 \theta
\]

\[
= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i (5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta)
\]

Then we equate the real and imaginary parts.

\[
\cos(5\theta) = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta
\]

\[
\sin(5\theta) = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta
\]

Finally we use the Pythagorean identity, \(\cos^2 \theta + \sin^2 \theta = 1\).

\[
\cos(5\theta) = \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2
\]

\[
\begin{align*}
\cos(5\theta) &= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta \\
\sin(5\theta) &= 5 (1 - \sin^2 \theta)^2 \sin \theta - 10 (1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta
\end{align*}
\]

\[
\begin{align*}
\sin(5\theta) &= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta
\end{align*}
\]

### 6.4 Arithmetic and Vectors

**Addition.** We can represent the complex number \(z = x + iy = re^{i\theta}\) as a vector in Cartesian space with tail at the origin and head at \((x, y)\), or equivalently, the vector of length \(r\) and angle \(\theta\). With the vector representation, we can add complex numbers by connecting the tail of one vector to the head of the other. The vector \(z + \zeta\) is the diagonal of the parallelogram defined by \(z\) and \(\zeta\). (See Figure 6.6.)

**Negation.** The negative of \(z = x + iy\) is \(-z = -x - iy\). In polar form we have \(z = re^{i\theta}\) and \(-z = re^{i(\theta + \pi)}\), (more generally, \(z = re^{i(\theta + (2n + 1)\pi)}\), \(n \in \mathbb{Z}\). In terms of vectors, \(-z\) has the same magnitude but opposite direction as \(z\). (See Figure 6.6.)
Multiplication. The product of \( z = re^{i\theta} \) and \( \zeta = \rho e^{i\phi} \) is \( z\zeta = r\rho e^{i(\theta + \phi)} \). The length of the vector \( z\zeta \) is the product of the lengths of \( z \) and \( \zeta \). The angle of \( z\zeta \) is the sum of the angles of \( z \) and \( \zeta \). (See Figure 6.6.)

Note that \( \arg(z\zeta) = \arg(z) + \arg(\zeta) \). Each of these arguments has an infinite number of values. If we write out the multi-valuedness explicitly, we have

\[
\{\theta + \phi + 2\pi n : n \in \mathbb{Z}\} = \{\theta + 2\pi n : n \in \mathbb{Z}\} + \{\phi + 2\pi n : n \in \mathbb{Z}\}
\]

The same is not true of the principal argument. In general, \( \text{Arg}(z\zeta) \neq \text{Arg}(z) + \text{Arg}(\zeta) \). Consider the case \( z = \zeta = e^{i\frac{3\pi}{4}} \). Then \( \text{Arg}(z) = \text{Arg}(\zeta) = \frac{3\pi}{4} \), however, \( \text{Arg}(z\zeta) = -\frac{\pi}{2} \).

![Figure 6.6: Addition, negation and multiplication.](image)

**Multiplicative inverse.** Assume that \( z \) is nonzero. The multiplicative inverse of \( z = re^{i\theta} \) is \( \frac{1}{z} = \frac{1}{r} e^{-i\theta} \). The length of \( \frac{1}{z} \) is the multiplicative inverse of the length of \( z \). The angle of \( \frac{1}{z} \) is the negative of the angle of \( z \). (See Figure 6.7.)
Division. Assume that $\zeta$ is nonzero. The quotient of $z = r e^{i\theta}$ and $\zeta = \rho e^{i\phi}$ is $\frac{z}{\zeta} = \frac{r}{\rho} e^{i(\theta-\phi)}$. The length of the vector $\frac{z}{\zeta}$ is the quotient of the lengths of $z$ and $\zeta$. The angle of $\frac{z}{\zeta}$ is the difference of the angles of $z$ and $\zeta$. (See Figure 6.7.)

Complex conjugate. The complex conjugate of $z = x + iy = r e^{i\theta}$ is $\bar{z} = x - iy = r e^{-i\theta}$. $\bar{z}$ is the mirror image of $z$, reflected across the $x$ axis. In other words, $\bar{z}$ has the same magnitude as $z$ and the angle of $\bar{z}$ is the negative of the angle of $z$. (See Figure 6.7.)

![Figure 6.7: Multiplicative inverse, division and complex conjugate.](image)

6.5 Integer Exponents

Consider the product $(a + b)^n$, $n \in \mathbb{Z}$. If we know $\arctan(a, b)$ then it will be most convenient to expand the product working in polar form. If not, we can write $n$ in base 2 to efficiently do the multiplications.
Example 6.5.1 Suppose that we want to write \((\sqrt{3} + i)^{20}\) in Cartesian form. We can do the multiplication directly. Note that 20 is 10100 in base 2. That is, \(20 = 2^4 + 2^2\). We first calculate the powers of the form \((\sqrt{3} + i)^{2n}\) by successive squaring.

\[
\begin{align*}
(\sqrt{3} + i)^2 &= 2 + i2\sqrt{3} \\
(\sqrt{3} + i)^4 &= -8 + i8\sqrt{3} \\
(\sqrt{3} + i)^8 &= -128 - i128\sqrt{3} \\
(\sqrt{3} + i)^{16} &= -32768 + i32768\sqrt{3}
\end{align*}
\]

Next we multiply \((\sqrt{3} + i)^4\) and \((\sqrt{3} + i)^{16}\) to obtain the answer.

\[
(\sqrt{3} + i)^{20} = \left(-32768 + i32768\sqrt{3}\right) \left(-8 + i8\sqrt{3}\right) = -524288 - i524288\sqrt{3}
\]

Since we know that \(\arctan(\sqrt{3}, 1) = \pi/6\), it is easiest to do this problem by first changing to modulus-argument form.

\[
(\sqrt{3} + i)^{20} = \left(\sqrt{\left(\sqrt{3}\right)^2 + 1^2} e^{i\arctan(\sqrt{3}, 1)}\right)^{20}
\]

\[
= (2 e^{i\pi/6})^{20}
\]

\[
= 2^{20} e^{i4\pi/3}
\]

\[
= 1048576 \left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right)
\]

\[
= -524288 - i524288\sqrt{3}
\]

\[\text{No, I have no idea why we would want to do that. Just humor me. If you pretend that you’re interested, I’ll do the same. Believe me, expressing your real feelings here isn’t going to do anyone any good.}\]
Example 6.5.2 Consider \((5 + i7)^{11}\). We will do the exponentiation in polar form and write the result in Cartesian form.

\[
(5 + i7)^{11} = \left(\sqrt{74} e^{i \arctan(5,7)}\right)^{11}
= 74^{5} \sqrt{74} \left(\cos(11 \arctan(5,7)) + i \sin(11 \arctan(5,7))\right)
= 2219006624 \sqrt{74} \cos(11 \arctan(5,7)) + i 2219006624 \sqrt{74} \sin(11 \arctan(5,7))
\]

The result is correct, but not very satisfying. This expression could be simplified. You could evaluate the trigonometric functions with some fairly messy trigonometric identities. This would take much more work than directly multiplying \((5 + i7)^{11}\).

6.6 Rational Exponents

In this section we consider complex numbers with rational exponents, \(z^{p/q}\), where \(p/q\) is a rational number. First we consider unity raised to the \(1/n\) power. We define \(1^{1/n}\) as the set of numbers \(\{z\}\) such that \(z^n = 1\).

\[
1^{1/n} = \{z \mid z^n = 1\}
\]

We can find these values by writing \(z\) in modulus-argument form.

\[
z^n = 1
r^n e^{in\theta} = 1
r^n = 1 \quad n\theta = 0 \mod 2\pi
r = 1 \quad \theta = 2\pi k \text{ for } k \in \mathbb{Z}
1^{1/n} = \{e^{i2\pi k/n} \mid k \in \mathbb{Z}\}
\]

There are only \(n\) distinct values as a result of the \(2\pi\) periodicity of \(e^{i\theta}\). \(e^{i2\pi} = e^0\).

\[
1^{1/n} = \{e^{i2\pi k/n} \mid k = 0, \ldots, n - 1\}
\]

These values are equally spaced points on the unit circle in the complex plane.
Example 6.6.1 \(1^{1/6}\) has the 6 values,

\[ \{ e^{i0}, e^{i\pi/3}, e^{2i\pi/3}, e^{i\pi}, e^{i4\pi/3}, e^{i5\pi/3} \} . \]

In Cartesian form this is

\[ \left\{ 1, \frac{1 + i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}, -1, \frac{-1 - i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2} \right\} . \]

The sixth roots of unity are plotted in Figure 6.8.

The sixth roots of unity are plotted in Figure 6.8.

![Figure 6.8: The sixth roots of unity.](image)

The \(n\)th roots of the complex number \(c = \alpha e^{i\beta}\) are the set of numbers \(z = r e^{i\theta}\) such that

\[ z^n = c = \alpha e^{i\beta} \]

\[ r^n e^{in\theta} = \alpha e^{i\beta} \]

\[ r = \sqrt[n]{\alpha} \quad n\theta = \beta \mod 2\pi \]

\[ r = \sqrt[n]{\alpha} \quad \theta = (\beta + 2\pi k)/n \text{ for } k = 0, \ldots, n - 1. \]

Thus

\[ c^{1/n} = \left\{ \sqrt[n]{\alpha} e^{i(\beta+2\pi k)/n} \mid k = 0, \ldots, n - 1 \right\} = \left\{ \frac{\sqrt[n]{|c|}}{e^{i(\text{Arg}(c)+2\pi k)/n}} \mid k = 0, \ldots, n - 1 \right\} \]
Principal roots. The principal $n^{\text{th}}$ root is denoted
\[ \sqrt[n]{z} \equiv e^{\text{Arg}(z)/n}. \]

Thus the principal root has the property
\[ -\pi/n < \text{Arg}(\sqrt[n]{z}) \leq \pi/n. \]

This is consistent with the notation from functions of a real variable: $\sqrt[n]{x}$ denotes the positive $n^{\text{th}}$ root of a positive real number. We adopt the convention that $z^{1/n}$ denotes the $n^{\text{th}}$ roots of $z$, which is a set of $n$ numbers and $\sqrt[n]{z}$ is the principal $n^{\text{th}}$ root of $z$, which is a single number. The $n^{\text{th}}$ roots of $z$ are the principal $n^{\text{th}}$ root of $z$ times the $n^{\text{th}}$ roots of unity.

\[
z^{1/n} = \left\{ e^{i\text{Arg}(z)+2\pi k/n} \mid k = 0, \ldots, n-1 \right\}
\]
\[
z^{1/n} = \left\{ e^{i2\pi k/n} \mid k = 0, \ldots, n-1 \right\}
\]
\[
z^{1/n} = \sqrt[n]{z}^{1/n}
\]

Rational exponents. We interpret $z^{p/q}$ to mean $z^{(p/q)}$. That is, we first simplify the exponent, i.e. reduce the fraction, before carrying out the exponentiation. Therefore $z^{2/4} = z^{1/2}$ and $z^{10/5} = z^2$. If $p/q$ is a reduced fraction, $(p$ and $q$ are relatively prime, in other words, they have no common factors), then
\[
z^{p/q} \equiv (z^p)^{1/q}.
\]

Thus $z^{p/q}$ is a set of $q$ values. Note that for an un-reduced fraction $r/s$,
\[
(z^r)^{1/s} \neq (z^{1/s})^r.
\]

The former expression is a set of $s$ values while the latter is a set of no more that $s$ values. For instance, $(1^2)^{1/2} = 1^{1/2} = \pm 1$ and $(1^{1/2})^2 = (\pm 1)^2 = 1$.

Example 6.6.2 Consider $2^{1/5}$, $(1 + i)^{1/3}$ and $(2 + i)^{5/6}$.
\[
2^{1/5} = \sqrt[5]{2} e^{i2\pi k/5}, \quad \text{for } k = 0, 1, 2, 3, 4
\]
\[
(1 + i)^{1/3} = \left( \sqrt{2} e^{i\pi/4} \right)^{1/3} \\
= \sqrt{2} e^{i\pi/12} e^{i2\pi k/3}, \quad \text{for } k = 0, 1, 2
\]

\[
(2 + i)^{5/6} = \left( \sqrt{5} e^{i \text{Arctan}(2, 1)} \right)^{5/6} \\
= \left( \sqrt{5} e^{i5 \text{Arctan}(2, 1)} \right)^{1/6} \\
= \frac{12}{5} e^{i5 \text{Arctan}(2, 1) e^{i\pi k/3}}, \quad \text{for } k = 0, 1, 2, 3, 4, 5
\]

**Example 6.6.3** We find the roots of \( z^5 + 4 \).

\[
(-4)^{1/5} = (4 e^{i\pi})^{1/5} \\
= \sqrt[5]{4} e^{i\pi(1+2k)/5}, \quad \text{for } k = 0, 1, 2, 3, 4
\]