Chapter 28

Fourier Series

Every time I close my eyes
The noise inside me amplifies
I can't escape
I relive every moment of the day
Every misstep I have made
Finds a way it can invade
My every thought
And this is why I find myself awake

-Failure
-Tom Shear (Assemblage 23)

28.1 An Eigenvalue Problem.

A self adjoint eigenvalue problem. Consider the eigenvalue problem

$$y'' + \lambda y = 0, \qquad y(-\pi) = y(\pi), \qquad y'(-\pi) = y'(\pi).$$

We rewrite the equation so the eigenvalue is on the right side.

$$L[y] \equiv -y'' = \lambda y$$

We demonstrate that this eigenvalue problem is self adjoint.

$$\begin{aligned}
\langle v|L[u]\rangle - \langle L[v]|u\rangle &= \langle v| - u''\rangle - \langle -v''|u\rangle \\
&= [-\bar{v}u']_{-\pi}^{\pi} + \langle v'|u'\rangle - [-\bar{v}'u]-\pi^{\pi} - \langle v'|u'\rangle \\
&= -\overline{v(\pi)}u'(\pi) + \overline{v(-\pi)}u'(-\pi) + \overline{v'(\pi)}u(\pi) - \overline{v'(-\pi)}u(-\pi) \\
&= -\overline{v(\pi)}u'(\pi) + \overline{v(\pi)}u'(\pi) + \overline{v'(\pi)}u(\pi) - \overline{v'(\pi)}u(\pi) \\
&= 0
\end{aligned}$$

Since Green's Identity reduces to $\langle v|L[u]\rangle - \langle L[v]|u\rangle = 0$, the problem is self adjoint. This means that the eigenvalues are real and that eigenfunctions corresponding to distinct eigenvalues are orthogonal. We compute the Rayleigh quotient for an eigenvalue λ with eigenfunction ϕ.

$$\begin{aligned}
\lambda &= \frac{-[\bar{\phi}\phi']_{-\pi}^{\pi} + \langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle} \\
&= \frac{-\overline{\phi(\pi)}\phi'(\pi) + \overline{\phi(-\pi)}\phi'(-\pi) + \langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle} \\
&= \frac{-\overline{\phi(\pi)}\phi'(\pi) + \overline{\phi(\pi)}\phi'(\pi) + \langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle} \\
&= \frac{\langle \phi'|\phi'\rangle}{\langle \phi|\phi\rangle}
\end{aligned}$$

We see that the eigenvalues are non-negative.

Computing the eigenvalues and eigenfunctions. Now we find the eigenvalues and eigenfunctions. First we consider the case $\lambda = 0$. The general solution of the differential equation is

$$y = c_1 + c_2 x.$$

The solution that satisfies the boundary conditions is $y = \text{const}$.

Now consider $\lambda > 0$. The general solution of the differential equation is

$$y = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right).$$

We apply the first boundary condition.

$$\begin{aligned}
y(-\pi) &= y(\pi) \\
c_1 \cos\left(-\sqrt{\lambda}\pi\right) + c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\
c_1 \cos\left(\sqrt{\lambda}\pi\right) - c_2 \sin\left(\sqrt{\lambda}\pi\right) &= c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\
c_2 \sin\left(\sqrt{\lambda}\pi\right) &= 0
\end{aligned}$$

Then we apply the second boundary condition.

$$\begin{aligned}
y'(-\pi) &= y'(\pi) \\
-c_1\sqrt{\lambda} \sin\left(-\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda} \cos\left(-\sqrt{\lambda}\pi\right) &= -c_1\sqrt{\lambda} \sin\left(\sqrt{\lambda}\pi\right) + c_2\sqrt{\lambda} \cos\left(\sqrt{\lambda}\pi\right) \\
c_1 \sin\left(\sqrt{\lambda}\pi\right) + c_2 \cos\left(\sqrt{\lambda}\pi\right) &= -c_1 \sin\left(\sqrt{\lambda}\pi\right) + c_2 \cos\left(\sqrt{\lambda}\pi\right) \\
c_1 \sin\left(\sqrt{\lambda}\pi\right) &= 0
\end{aligned}$$

To satisify the two boundary conditions either $c_1 = c_2 = 0$ or $\sin\left(\sqrt{\lambda}\pi\right) = 0$. The former yields the trivial solution. The latter gives us the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{Z}^+$. The corresponding solution is

$$y_n = c_1 \cos(nx) + c_2 \sin(nx).$$

There are two eigenfunctions for each of the positive eigenvalues.

We choose the eigenvalues and eigenfunctions.

$$\begin{aligned}
&\lambda_0 = 0, && \phi_0 = \frac{1}{2} \\
&\lambda_n = n^2, && \phi_{2n-1} = \cos(nx), \quad \phi_{2n} = \sin(nx), \quad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

Orthogonality of Eigenfunctions. We know that the eigenfunctions of distinct eigenvalues are orthogonal. In addition, the two eigenfunctions of each positive eigenvalue are orthogonal.

$$\int_{-\pi}^{\pi} \cos(nx)\sin(nx)\,\mathrm{d}x = \left[\frac{1}{2n}\sin^2(nx)\right]_{-\pi}^{\pi} = 0$$

Thus the eigenfunctions $\{\frac{1}{2}, \cos(x), \sin(x), \cos(2x), \sin(2x)\}$ are an orthogonal set.

28.2 Fourier Series.

A series of the eigenfunctions

$$\phi_0 = \frac{1}{2}, \quad \phi_n^{(1)} = \cos(nx), \quad \phi_n^{(2)} = \sin(nx), \quad \text{for } n \geq 1$$

is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big).$$

This is known as a *Fourier series*. (We choose $\phi_0 = \frac{1}{2}$ so all of the eigenfunctions have the same norm.) A fairly general class of functions can be expanded in Fourier series. Let $f(x)$ be a function defined on $-\pi < x < \pi$. Assume that $f(x)$ can be expanded in a Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big). \tag{28.1}$$

Here the "$\sim$" means "has the Fourier series". We have not said if the series converges yet. For now let's assume that the series converges uniformly so we can replace the $\sim$ with an $=$.

We integrate Equation 28.1 from $-\pi$ to π to determine a_0.

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \mathrm{d}x + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)\,\mathrm{d}x$$

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \pi a_0 + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x \right)$$

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \pi a_0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

Multiplying by $\cos(mx)$ and integrating will enable us to solve for a_m.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x \right)$$

All but one of the terms on the right side vanishes due to the orthogonality of the eigenfunctions.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = a_m \int_{-\pi}^{\pi} \cos(mx)\cos(mx)\,\mathrm{d}x$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = a_m \int_{-\pi}^{\pi} \left(\frac{1}{2} + \cos(2mx) \right)\,\mathrm{d}x$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \pi a_m$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x.$$

Note that this formula is valid for $m = 0, 1, 2, \ldots$.

Similarly, we can multiply by $\sin(mx)$ and integrate to solve for b_m. The result is

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, \mathrm{d}x.$$

a_n and b_n are called *Fourier coefficients*.

Although we will not show it, Fourier series converge for a fairly general class of functions. Let $f(x^-)$ denote the left limit of $f(x)$ and $f(x^+)$ denote the right limit.

Example 28.2.1 *For the function defined*

$$f(x) = \begin{cases} 0 & \text{for } x < 0, \\ x + 1 & \text{for } x \geq 0, \end{cases}$$

the left and right limits at $x = 0$ are

$$f(0^-) = 0, \qquad f(0^+) = 1.$$

Result 28.2.1 Let $f(x)$ be a 2π-periodic function for which $\int_{-\pi}^{\pi} |f(x)| \, \mathrm{d}x$ exists. Define the Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

If x is an interior point of an interval on which $f(x)$ has limited total fluctuation, then the Fourier series of $f(x)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big),$$

converges to $\frac{1}{2}(f(x^-) + f(x^+))$. If f is continuous at x, then the series converges to $f(x)$.

Periodic Extension of a Function. Let $g(x)$ be a function that is arbitrarily defined on $-\pi \leq x < \pi$. The Fourier series of $g(x)$ will represent the periodic extension of $g(x)$. The periodic extension, $f(x)$, is defined by the two conditions:

$$f(x) = g(x) \quad \text{for } -\pi \leq x < \pi,$$
$$f(x + 2\pi) = f(x).$$

The periodic extension of $g(x) = x^2$ is shown in Figure 28.1.

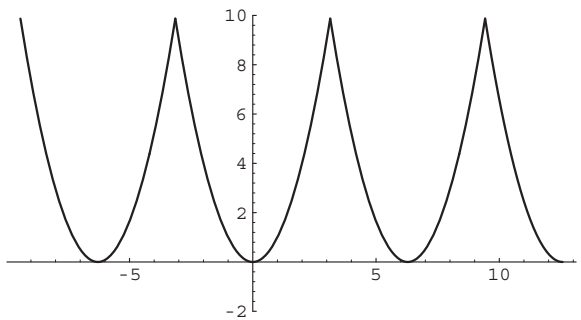

Figure 28.1: The Periodic Extension of $g(x) = x^2$.

Limited Fluctuation. A function that has limited total fluctuation can be written $f(x) = \psi_+(x) - \psi_-(x)$, where ψ_+ and ψ_- are bounded, nondecreasing functions. An example of a function that does not have limited total fluctuation

is $\sin(1/x)$, whose fluctuation is unlimited at the point $x = 0$.

Functions with Jump Discontinuities. Let $f(x)$ be a discontinuous function that has a convergent Fourier series. Note that the series does not necessarily converge to $f(x)$. Instead it converges to $\hat{f}(x) = \frac{1}{2}(f(x^-) + f(x^+))$.

Example 28.2.2 *Consider the function defined by*

$$f(x) = \begin{cases} -x & \text{for } -\pi \le x < 0 \\ \pi - 2x & \text{for } 0 \le x < \pi. \end{cases}$$

The Fourier series converges to the function defined by

$$\hat{f}(x) = \begin{cases} 0 & \text{for } x = -\pi \\ -x & \text{for } -\pi < x < 0 \\ \pi/2 & \text{for } x = 0 \\ \pi - 2x & \text{for } 0 < x < \pi. \end{cases}$$

The function $\hat{f}(x)$ is plotted in Figure 28.2.

28.3 Least Squares Fit

Approximating a function with a Fourier series. Suppose we want to approximate a 2π-periodic function $f(x)$ with a finite Fourier series.

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

Here the coefficients are computed with the familiar formulas. Is this the best approximation to the function? That is, is it possible to choose coefficients α_n and β_n such that

$$f(x) \approx \frac{\alpha_0}{2} + \sum_{n=1}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx))$$

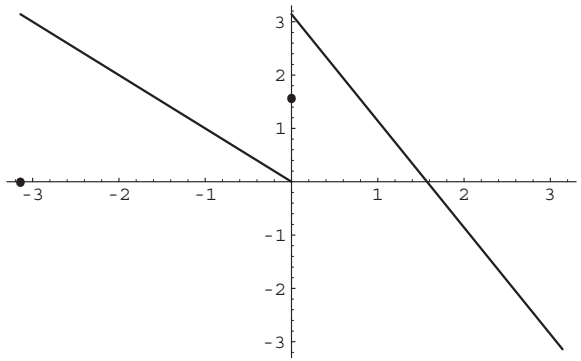

Figure 28.2: Graph of $\hat{f}(x)$.

would give a better approximation?

Least squared error fit. The most common criterion for finding the best fit to a function is the least squares fit. The best approximation to a function is defined as the one that minimizes the integral of the square of the deviation. Thus if $f(x)$ is to be approximated on the interval $a \leq x \leq b$ by a series

$$f(x) \approx \sum_{n=1}^{N} c_n \phi_n(x), \tag{28.2}$$

the best approximation is found by choosing values of c_n that minimize the error E.

$$E \equiv \int_a^b \left| f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right|^2 \, \mathrm{d}x$$

Generalized Fourier coefficients. We consider the case that the ϕ_n are orthogonal. For simplicity, we also assume that the ϕ_n are real-valued. Then most of the terms will vanish when we interchange the order of integration and summation.

$$E = \int_a^b \left(f^2 - 2f \sum_{n=1}^{N} c_n \phi_n + \sum_{n=1}^{N} c_n \phi_n \sum_{m=1}^{N} c_m \phi_m \right) \mathrm{d}x$$

$$E = \int_a^b f^2 \, \mathrm{d}x - 2 \sum_{n=1}^{N} c_n \int_a^b f \phi_n \, \mathrm{d}x + \sum_{n=1}^{N} \sum_{m=1}^{N} c_n c_m \int_a^b \phi_n \phi_m \, \mathrm{d}x$$

$$E = \int_a^b f^2 \, \mathrm{d}x - 2 \sum_{n=1}^{N} c_n \int_a^b f \phi_n \, \mathrm{d}x + \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2 \, \mathrm{d}x$$

$$E = \int_a^b f^2 \, \mathrm{d}x + \sum_{n=1}^{N} \left(c_n^2 \int_a^b \phi_n^2 \, \mathrm{d}x - 2c_n \int_a^b f \phi_n \, \mathrm{d}x \right)$$

We complete the square for each term.

$$E = \int_a^b f^2 \, \mathrm{d}x + \sum_{n=1}^{N} \left(\int_a^b \phi_n^2 \, \mathrm{d}x \left(c_n - \frac{\int_a^b f \phi_n \, \mathrm{d}x}{\int_a^b \phi_n^2 \, \mathrm{d}x} \right)^2 - \left(\frac{\int_a^b f \phi_n \, \mathrm{d}x}{\int_a^b \phi_n^2 \, \mathrm{d}x} \right)^2 \right)$$

Each term involving c_n is non-negative, and is minimized for

$$c_n = \frac{\int_a^b f \phi_n \, \mathrm{d}x}{\int_a^b \phi_n^2 \, \mathrm{d}x}. \tag{28.3}$$

We call these the *generalized Fourier coefficients*.

For such a choice of the c_n, the error is

$$E = \int_a^b f^2 \, \mathrm{d}x - \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2 \, \mathrm{d}x.$$

Since the error is non-negative, we have

$$\int_a^b f^2 \, \mathrm{d}x \geq \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2 \, \mathrm{d}x.$$

This is known as *Bessel's Inequality*. If the series in Equation 28.2 converges in the mean to $f(x)$, $\lim N \to \infty E = 0$, then we have equality as $N \to \infty$.

$$\int_a^b f^2 \, \mathrm{d}x = \sum_{n=1}^{\infty} c_n^2 \int_a^b \phi_n^2 \, \mathrm{d}x.$$

This is *Parseval's equality*.

Fourier coefficients. Previously we showed that if the series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx),$$

converges uniformly then the coefficients in the series are the Fourier coefficients,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

Now we show that by choosing the coefficients to minimize the squared error, we obtain the same result. We apply Equation 28.3 to the Fourier eigenfunctions.

$$
\begin{aligned}
a_0 &= \frac{\int_{-\pi}^{\pi} f \frac{1}{2}\,\mathrm{d}x}{\int_{-\pi}^{\pi} \frac{1}{4}\,\mathrm{d}x} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\
a_n &= \frac{\int_{-\pi}^{\pi} f\cos(nx)\,\mathrm{d}x}{\int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
b_n &= \frac{\int_{-\pi}^{\pi} f\sin(nx)\,\mathrm{d}x}{\int_{-\pi}^{\pi} \sin^2(nx)\,\mathrm{d}x} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x
\end{aligned}
$$

28.4 Fourier Series for Functions Defined on Arbitrary Ranges

If $f(x)$ is defined on $c-d \le x < c+d$ and $f(x+2d) = f(x)$, then $f(x)$ has a Fourier series of the form

$$
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi(x+c)}{d}\right) + b_n \sin\left(\frac{n\pi(x+c)}{d}\right).
$$

Since

$$
\int_{c-d}^{c+d} \cos^2\left(\frac{n\pi(x+c)}{d}\right)\,\mathrm{d}x = \int_{c-d}^{c+d} \sin^2\left(\frac{n\pi(x+c)}{d}\right)\,\mathrm{d}x = d,
$$

the Fourier coefficients are given by the formulas

$$
\begin{aligned}
a_n &= \frac{1}{d}\int_{c-d}^{c+d} f(x)\cos\left(\frac{n\pi(x+c)}{d}\right)\,\mathrm{d}x \\
b_n &= \frac{1}{d}\int_{c-d}^{c+d} f(x)\sin\left(\frac{n\pi(x+c)}{d}\right)\,\mathrm{d}x.
\end{aligned}
$$

Example 28.4.1 *Consider the function defined by*

$$f(x) = \begin{cases} x+1 & \text{for } -1 \le x < 0 \\ x & \text{for } 0 \le x < 1 \\ 3-2x & \text{for } 1 \le x < 2. \end{cases}$$

This function is graphed in Figure 28.3.

The Fourier series converges to $\hat{f}(x) = (f(x^-) + f(x^+))/2$,

$$\hat{f}(x) = \begin{cases} -\frac{1}{2} & \text{for } x = -1 \\ x+1 & \text{for } -1 < x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ x & \text{for } 0 < x < 1 \\ 3-2x & \text{for } 1 \le x < 2. \end{cases}$$

$\hat{f}(x)$ *is also graphed in Figure 28.3.*

The Fourier coefficients are

$$\begin{aligned} a_n &= \frac{1}{3/2}\int_{-1}^{2} f(x)\cos\left(\frac{2n\pi(x+1/2)}{3}\right)\,\mathrm{d}x \\ &= \frac{2}{3}\int_{-1/2}^{5/2} f(x-1/2)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\ &= \frac{2}{3}\int_{-1/2}^{1/2}(x+1/2)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x + \frac{2}{3}\int_{1/2}^{3/2}(x-1/2)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\ &\qquad + \frac{2}{3}\int_{3/2}^{5/2}(4-2x)\cos\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\ &= -\frac{1}{(n\pi)^2}\sin\left(\frac{2n\pi}{3}\right)\left[2(-1)^n n\pi + 9\sin\left(\frac{n\pi}{3}\right)\right] \end{aligned}$$

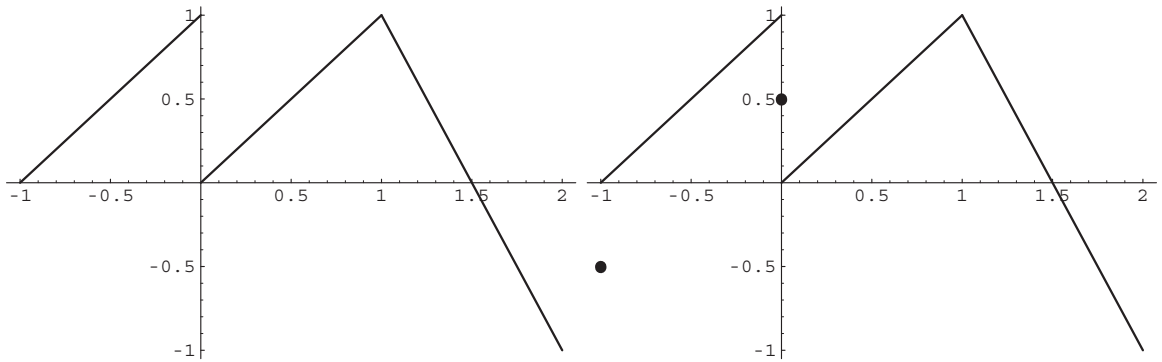

Figure 28.3: A Function Defined on the range $-1 \leq x < 2$ and the Function to which the Fourier Series Converges.

$$
\begin{aligned}
b_n &= \frac{1}{3/2}\int_{-1}^{2} f(x)\sin\left(\frac{2n\pi(x+1/2)}{3}\right)\,\mathrm{d}x \\
&= \frac{2}{3}\int_{-1/2}^{5/2} f(x-1/2)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&= \frac{2}{3}\int_{-1/2}^{1/2}(x+1/2)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x + \frac{2}{3}\int_{1/2}^{3/2}(x-1/2)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&\qquad + \frac{2}{3}\int_{3/2}^{5/2}(4-2x)\sin\left(\frac{2n\pi x}{3}\right)\,\mathrm{d}x \\
&= -\frac{2}{(n\pi)^2}\sin^2\left(\frac{n\pi}{3}\right)\left[2(-1)^n n\pi + 4n\pi\cos\left(\frac{n\pi}{3}\right) - 3\sin\left(\frac{n\pi}{3}\right)\right]
\end{aligned}
$$

28.5 Fourier Cosine Series

If $f(x)$ is an even function, $(f(-x) = f(x))$, then there will not be any sine terms in the Fourier series for $f(x)$. The Fourier sine coefficient is

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, \mathrm{d}x.$$

Since $f(x)$ is an even function and $\sin(nx)$ is odd, $f(x)\sin(nx)$ is odd. b_n is the integral of an odd function from $-\pi$ to π and is thus zero. We can rewrite the cosine coefficients,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx) \, \mathrm{d}x. \end{aligned}$$

Example 28.5.1 *Consider the function defined on* $[0, \pi)$ *by*

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi/2 \\ \pi - x & \text{for } \pi/2 \le x < \pi. \end{cases}$$

The Fourier cosine coefficients for this function are

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{0}^{\pi/2} x \cos(nx) \, \mathrm{d}x + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cos(nx) \, \mathrm{d}x \\ &= \begin{cases} \frac{\pi}{4} & \text{for } n = 0, \\ \frac{8}{\pi n^2} \cos\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{n\pi}{4}\right) & \text{for } n \ge 1. \end{cases} \end{aligned}$$

In Figure 28.4 the even periodic extension of $f(x)$ is plotted in a dashed line and the sum of the first five nonzero terms in the Fourier cosine series are plotted in a solid line.

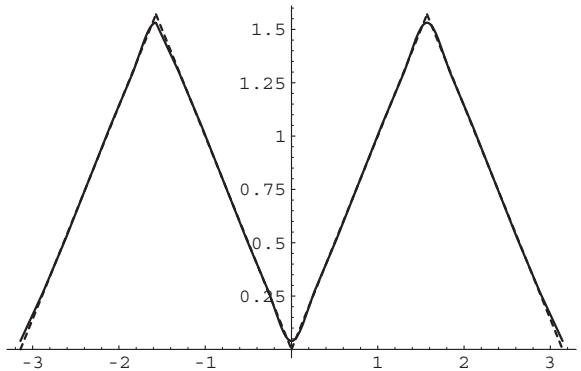

Figure 28.4: Fourier Cosine Series.

28.6 Fourier Sine Series

If $f(x)$ is an odd function, $(f(-x) = -f(x))$, then there will not be any cosine terms in the Fourier series. Since $f(x)\cos(nx)$ is an odd function, the cosine coefficients will be zero. Since $f(x)\sin(nx)$ is an even function,we can rewrite the sine coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

Example 28.6.1 *Consider the function defined on* $[0, \pi)$ *by*

$$f(x) = \begin{cases} x & \text{for } 0 \le x < \pi/2 \\ \pi - x & \text{for } \pi/2 \le x < \pi. \end{cases}$$

The Fourier sine coefficients for this function are

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,\mathrm{d}x \\ &= \frac{16}{\pi n^2}\cos\left(\frac{n\pi}{4}\right)\sin^3\left(\frac{n\pi}{4}\right) \end{aligned}$$

In Figure 28.5 the odd periodic extension of $f(x)$ *is plotted in a dashed line and the sum of the first five nonzero terms in the Fourier sine series are plotted in a solid line.*

28.7 Complex Fourier Series and Parseval's Theorem

By writing $\sin(nx)$ and $\cos(nx)$ in terms of $\mathrm{e}^{\imath nx}$ and $\mathrm{e}^{-\imath nx}$ we can obtain the complex form for a Fourier series.

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\frac{1}{2}(\mathrm{e}^{\imath nx} + \mathrm{e}^{-\imath nx}) + b_n\frac{1}{\imath 2}(\mathrm{e}^{\imath nx} - \mathrm{e}^{-\imath nx})\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{2}(a_n - \imath b_n)\,\mathrm{e}^{\imath nx} + \frac{1}{2}(a_n + \imath b_n)\,\mathrm{e}^{-\imath nx}\right) \\ &= \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\imath nx} \end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{2}(a_n - \imath b_n) & \text{for } n \ge 1 \\ \frac{a_0}{2} & \text{for } n = 0 \\ \frac{1}{2}(a_{-n} + \imath b_{-n}) & \text{for } n \le -1. \end{cases}$$

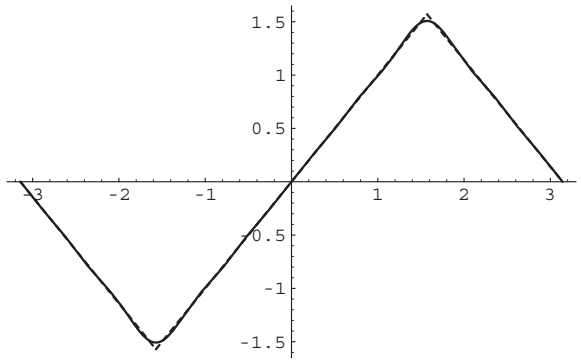

Figure 28.5: Fourier Sine Series.

The functions $\{\ldots, \mathrm{e}^{-\imath x}, 1, \mathrm{e}^{\imath x}, \mathrm{e}^{\imath 2x}, \ldots\}$, satisfy the relation

$$\int_{-\pi}^{\pi} \mathrm{e}^{\imath nx}\, \mathrm{e}^{-\imath mx}\, \mathrm{d}x = \int_{-\pi}^{\pi} \mathrm{e}^{\imath (n-m)x}\, \mathrm{d}x$$
$$= \begin{cases} 2\pi & \text{for } n = m \\ 0 & \text{for } n \neq m. \end{cases}$$

Starting with the complex form of the Fourier series of a function $f(x)$,

$$f(x) \sim \sum_{-\infty}^{\infty} c_n\, \mathrm{e}^{\imath nx},$$

we multiply by $\mathrm{e}^{-\imath mx}$ and integrate from $-\pi$ to π to obtain

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\imath mx}\,\mathrm{d}x = \int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} c_n\,\mathrm{e}^{\imath nx}\,\mathrm{e}^{-\imath mx}\,\mathrm{d}x$$

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\imath mx}\,\mathrm{d}x$$

If $f(x)$ is real-valued then

$$c_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{\imath mx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{(\mathrm{e}^{-\imath mx})}\,\mathrm{d}x = \overline{c_m}$$

where $\bar{z}$ denotes the complex conjugate of z.

Assume that $f(x)$ has a uniformly convergent Fourier series.

$$\begin{aligned}
\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} c_m\,\mathrm{e}^{\imath mx}\right)\left(\sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\imath nx}\right)\mathrm{d}x \\
&= 2\pi\sum_{n=-\infty}^{\infty} c_n c_{-n} \\
&= 2\pi\left(\sum_{n=-\infty}^{-1}\left[\frac{1}{4}(a_{-n}+\imath b_{-n})(a_{-n}-\imath b_{-n})\right] + \frac{a_0}{2}\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{4}(a_n-\imath b_n)(a_n+\imath b_n)\right]\right) \\
&= 2\pi\left(\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right)
\end{aligned}$$

This yields a result known as Parseval's theorem which holds even when the Fourier series of $f(x)$ is not uniformly convergent.

Result 28.7.1 Parseval's Theorem. If $f(x)$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)),$$

then

$$\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{\pi}{2}a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

28.8 Behavior of Fourier Coefficients

Before we jump hip-deep into the grunge involved in determining the behavior of the Fourier coefficients, let's take a step back and get some perspective on what we should be looking for.

One of the important questions is whether the Fourier series converges uniformly. From Result 12.2.1 we know that a uniformly convergent series represents a continuous function. Thus we know that the Fourier series of a discontinuous function cannot be uniformly convergent. From Section 12.2 we know that a series is uniformly convergent if it can be bounded by a series of positive terms. If the Fourier coefficients, a_n and b_n, are $O(1/n^\alpha)$ where $\alpha > 1$ then the series can be bounded by (const) $\sum_{n=1}^{\infty} 1/n^\alpha$ and will thus be uniformly convergent.

Let $f(x)$ be a function that meets the conditions for having a Fourier series and in addition is bounded. Let $(-\pi, p_1), (p_1, p_2), (p_2, p_3), \ldots, (p_m, \pi)$ be a partition into a finite number of intervals of the domain, $(-\pi, \pi)$ such that on each interval $f(x)$ and all it's derivatives are continuous. Let $f(p^-)$ denote the left limit of $f(p)$ and $f(p^+)$ denote the right limit.

$$f(p^-) = \lim_{\epsilon\to 0^+} f(p-\epsilon), \qquad f(p^+) = \lim_{\epsilon\to 0^+} f(p+\epsilon)$$

Example 28.8.1 *The function shown in Figure 28.6 would be partitioned into the intervals*

$$(-2,-1), (-1,0), (0,1), (1,2).$$

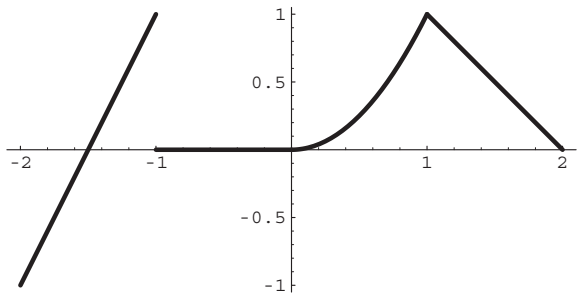

Figure 28.6: A Function that can be Partitioned.

Suppose $f(x)$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

We can use the integral formula to find the a_n's.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \\ &= \frac{1}{\pi} \left(\int_{-\pi}^{p_1} f(x) \cos(nx) \, \mathrm{d}x + \int_{p_1}^{p_2} f(x) \cos(nx) \, \mathrm{d}x + \cdots + \int_{p_m}^{\pi} f(x) \cos(nx) \, \mathrm{d}x \right) \end{aligned}$$

Using integration by parts,

$$
\begin{aligned}
&= \frac{1}{n\pi}\left(\Big[f(x)\sin(nx)\Big]_{-\pi}^{p_1} + \Big[f(x)\sin(nx)\Big]_{p_1}^{p_2} + \cdots + \Big[f(x)\sin(nx)\Big]_{p_m}^{\pi}\right) \\
&\qquad - \frac{1}{n\pi}\left(\int_{-\pi}^{p_1} f'(x)\sin(nx)\,\mathrm{d}x + \int_{p_1}^{p_2} f'(x)\sin(nx)\,\mathrm{d}x + \int_{p_m}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{1}{n\pi}\Big\{\big[f(p_1^-) - f(p_1^+)\big]\sin(np_1) + \cdots + \big[f(p_m^-) - f(p_m^+)\big]\sin(np_m)\Big\} \\
&\qquad - \frac{1}{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{n}A_n - \frac{1}{n}b_n'
\end{aligned}
$$

where

$$
A_n = \frac{1}{\pi}\sum_{j=1}^{m}\sin(np_j)\big[f(p_j^-) - f(p_j^+)\big]
$$

and the b_n' are the sine coefficients of $f'(x)$.

Since $f(x)$ is bounded, $A_n = O(1)$. Since $f'(x)$ is bounded,

$$
b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x = O(1).
$$

Thus $a_n = O(1/n)$ as $n \to \infty$. (Actually, from the Riemann-Lebesgue Lemma, $b_n' = \mathcal{O}(1/n)$.)

Now we repeat this analysis for the sine coefficients.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{p_1} f(x)\sin(nx)\,\mathrm{d}x + \int_{p_1}^{p_2} f(x)\sin(nx)\,\mathrm{d}x + \cdots + \int_{p_m}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right) \\
&= \frac{-1}{n\pi}\Big\{\big[f(x)\cos(nx)\big]_{-\pi}^{p_1} + \big[f(x)\cos(nx)\big]_{p_1}^{p_2} + \cdots + \big[f(x)\cos(nx)\big]_{p_m}^{\pi}\Big\} \\
&\qquad + \frac{1}{n\pi}\left(\int_{-\pi}^{p_1} f'(x)\cos(nx)\,\mathrm{d}x + \int_{p_1}^{p_2} f'(x)\cos(nx)\,\mathrm{d}x + \int_{p_m}^{\pi} f'(x)\cos(nx)\,\mathrm{d}x\right) \\
&= -\frac{1}{n}B_n + \frac{1}{n}a'_n
\end{aligned}
$$

where

$$
B_n = \frac{(-1)^n}{\pi}\big[f(-\pi) - f(\pi)\big] - \frac{1}{\pi}\sum_{j=1}^{m}\cos(np_j)\big[f(p_j^-) - f(p_j^+)\big]
$$

and the a'_n are the cosine coefficients of $f'(x)$.

Since $f(x)$ and $f'(x)$ are bounded, $B_n, a'_n = O(1)$ and thus $b_n = O(1/n)$ as $n \to \infty$.

With integration by parts on the Fourier coefficients of $f'(x)$ we could find that

$$
a'_n = \frac{1}{n}A'_n - \frac{1}{n}b''_n
$$

where $A'_n = \frac{1}{\pi}\sum_{j=1}^{m}\sin(np_j)[f'(p_j^-) - f'(p_j^+)]$ and the b''_n are the sine coefficients of $f''(x)$, and

$$
b'_n = -\frac{1}{n}B'_n + \frac{1}{n}a''_n
$$

where $B'_n = \frac{(-1)^n}{\pi}[f'(-\pi) - f'(\pi)] - \frac{1}{\pi}\sum_{j=1}^{m}\cos(np_j)[f'(p_j^-) - f'(p_j^+)]$ and the a''_n are the cosine coefficients of $f''(x)$.

Now we can rewrite a_n and b_n as

$$\begin{aligned} a_n &= \frac{1}{n}A_n + \frac{1}{n^2}B'_n - \frac{1}{n^2}a''_n \\ b_n &= -\frac{1}{n}B_n + \frac{1}{n^2}A'_n - \frac{1}{n^2}b''_n. \end{aligned}$$

Continuing this process we could define $A_n^{(j)}$ and $B_n^{(j)}$ so that

$$\begin{aligned} a_n &= \frac{1}{n}A_n + \frac{1}{n^2}B'_n - \frac{1}{n^3}A''_n - \frac{1}{n^4}B'''_n + \cdots \\ b_n &= -\frac{1}{n}B_n + \frac{1}{n^2}A'_n + \frac{1}{n^3}B''_n - \frac{1}{n^4}A'''_n - \cdots . \end{aligned}$$

For any bounded function, the Fourier coefficients satisfy $a_n, b_n = O(1/n)$ as $n \to \infty$. If A_n and B_n are zero then the Fourier coefficients will be $O(1/n^2)$. A sufficient condition for this is that the periodic extension of $f(x)$ is continuous. We see that if the periodic extension of $f'(x)$ is continuous then A'_n and B'_n will be zero and the Fourier coefficients will be $O(1/n^3)$.

Result 28.8.1 Let $f(x)$ be a bounded function for which there is a partition of the range $(-\pi, \pi)$ into a finite number of intervals such that $f(x)$ and all it's derivatives are continuous on each of the intervals. If $f(x)$ is not continuous then the Fourier coefficients are $O(1/n)$. If $f(x), f'(x), \ldots, f^{(k-2)}(x)$ are continuous then the Fourier coefficients are $O(1/n^k)$.

If the periodic extension of $f(x)$ is continuous, then the Fourier coefficients will be $O(1/n^2)$. The series $\sum_{n=1}^{\infty} |a_n \cos(nx)$ can be bounded by $M \sum_{n=1}^{\infty} 1/n^2$ where $M = \max_n(|a_n| + |b_n|)$. Thus the Fourier series converges to $f(x)$ uniformly.

Result 28.8.2 If the periodic extension of $f(x)$ is continuous then the Fourier series of $f(x)$ will converge uniformly for all x.

If the periodic extension of $f(x)$ is not continuous, we have the following result.

Result 28.8.3 If $f(x)$ is continuous in the interval $c < x < d$, then the Fourier series is uniformly convergent in the interval $c + \delta \leq x \leq d - \delta$ for any $\delta > 0$.

Example 28.8.2 Different Rates of Convergence.

A Discontinuous Function. *Consider the function defined by*

$$f_1(x) = \begin{cases} -1 & \text{for } -1 < x < 0 \\ 1, & \text{for } 0 < x < 1. \end{cases}$$

This function has jump discontinuities, so we know that the Fourier coefficients are $O(1/n)$.

Since this function is odd, there will only be sine terms in it's Fourier expansion. Furthermore, since the function is symmetric about $x = 1/2$, *there will be only odd sine terms. Computing these terms,*

$$\begin{aligned} b_n &= 2\int_0^1 \sin(n\pi x)\,\mathrm{d}x \\ &= 2\left[\frac{-1}{n\pi}\cos(n\pi x)\right]_0^1 \\ &= 2\left(-\frac{(-1)^n}{n\pi} - \frac{-1}{n\pi}\right) \\ &= \begin{cases} \frac{4}{n\pi} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases} \end{aligned}$$

The function and the sum of the first three terms in the expansion are plotted, in dashed and solid lines respectively, in Figure 28.7. Although the three term sum follows the general shape of the function, it is clearly not a good approximation.

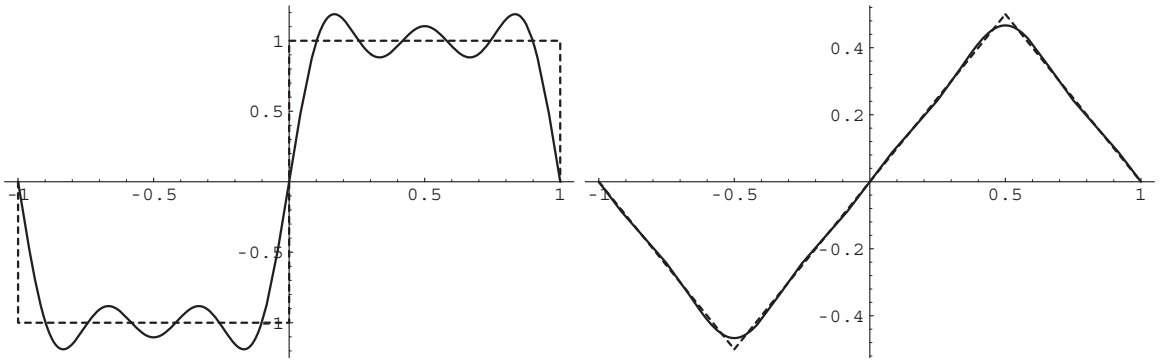

Figure 28.7: Three Term Approximation for a Function with Jump Discontinuities and a Continuous Function.

A Continuous Function. *Consider the function defined by*

$$f_2(x) = \begin{cases} -x-1 & \textit{for } -1 < x < -1/2 \\ x & \textit{for } -1/2 < x < 1/2 \\ -x+1 & \textit{for } 1/2 < x < 1. \end{cases}$$

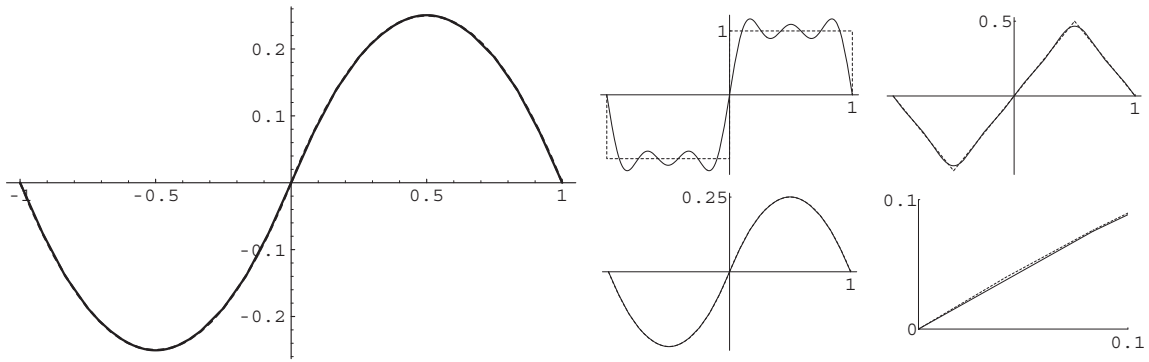

Figure 28.8: Three Term Approximation for a Function with Continuous First Derivative and Comparison of the Rates of Convergence.

Since this function is continuous, the Fourier coefficients will be $O(1/n^2)$. Also we see that there will only be odd sine terms in the expansion.

$$
\begin{aligned}
b_n &= \int_{-1}^{-1/2} (-x-1)\sin(n\pi x)\,\mathrm{d}x + \int_{-1/2}^{1/2} x\sin(n\pi x)\,\mathrm{d}x + \int_{1/2}^{1} (-x+1)\sin(n\pi x)\,\mathrm{d}x \\
&= 2\int_{0}^{1/2} x\sin(n\pi x)\,\mathrm{d}x + 2\int_{1/2}^{1} (1-x)\sin(n\pi x)\,\mathrm{d}x \\
&= \frac{4}{(n\pi)^2}\sin(n\pi/2) \\
&= \begin{cases} \frac{4}{(n\pi)^2}(-1)^{(n-1)/2} & \textit{for odd } n \\ 0 & \textit{for even } n. \end{cases}
\end{aligned}
$$

The function and the sum of the first three terms in the expansion are plotted, in dashed and solid lines respectively, in Figure 28.7. We see that the convergence is much better than for the function with jump discontinuities.

A Function with a Continuous First Derivative. *Consider the function defined by*

$$f_3(x) = \begin{cases} x(1+x) & \text{for } -1 < x < 0 \\ x(1-x) & \text{for } 0 < x < 1. \end{cases}$$

Since the periodic extension of this function is continuous and has a continuous first derivative, the Fourier coefficients will be $O(1/n^3)$. We see that the Fourier expansion will contain only odd sine terms.

$$\begin{aligned} b_n &= \int_{-1}^{0} x(1+x)\sin(n\pi x)\,\mathrm{d}x + \int_{0}^{1} x(1-x)\sin(n\pi x)\,\mathrm{d}x \\ &= 2\int_{0}^{1} x(1-x)\sin(n\pi x)\,\mathrm{d}x \\ &= \frac{4(1-(-1)^n)}{(n\pi)^3} \\ &= \begin{cases} \frac{4}{(n\pi)^3} & \text{for odd } n \\ 0 & \text{for even } n. \end{cases} \end{aligned}$$

The function and the sum of the first three terms in the expansion are plotted in Figure 28.8. We see that the first three terms give a very good approximation to the function. The plots of the function, (in a dashed line), and the three term approximation, (in a solid line), are almost indistinguishable.

In Figure 28.8 the convergence of the of the first three terms to $f_1(x)$, $f_2(x)$, and $f_3(x)$ are compared. In the last graph we see a closeup of $f_3(x)$ and it's Fourier expansion to show the error.

28.9 Gibb's Phenomenon

The Fourier expansion of

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ -1 & \text{for } -1 \le x < 0 \end{cases}$$

is

$$f(x) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x).$$

For any fixed x, the series converges to $\frac{1}{2}(f(x^-) + f(x^+))$. For any $\delta > 0$, the convergence is uniform in the intervals $-1 + \delta \le x \le -\delta$ and $\delta \le x \le 1 - \delta$. How will the nonuniform convergence at integral values of x affect the Fourier series? Finite Fourier series are plotted in Figure 28.9 for 5, 10, 50 and 100 terms. (The plot for 100 terms is closeup of the behavior near $x = 0$.) Note that at each discontinuous point there is a series of overshoots and undershoots that are pushed closer to the discontinuity by increasing the number of terms, but do not seem to decrease in height. In fact, as the number of terms goes to infinity, the height of the overshoots and undershoots does not vanish. This is known as Gibb's phenomenon.

28.10 Integrating and Differentiating Fourier Series

Integrating Fourier Series. Since integration is a smoothing operation, any convergent Fourier series can be integrated term by term to yield another convergent Fourier series.

Example 28.10.1 *Consider the step function*

$$f(x) = \begin{cases} \pi & \textit{for } 0 \le x < \pi \\ -\pi & \textit{for } -\pi \le x < 0. \end{cases}$$

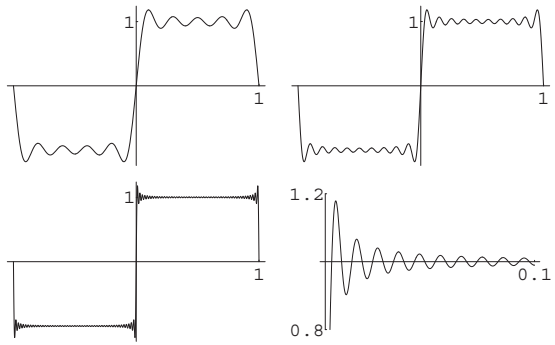

Figure 28.9:

Since this is an odd function, there are no cosine terms in the Fourier series.

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \pi \sin(nx)\,\mathrm{d}x \\
&= 2\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\
&= \frac{2}{n}(1-(-1)^n) \\
&= \begin{cases} \frac{4}{n} & \textit{for odd } n \\ 0 & \textit{for even } n. \end{cases}
\end{aligned}
$$

$$f(x) \sim \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{4}{n} \sin nx$$

Integrating this relation,

$$\begin{aligned}
\int_{-\pi}^{x} f(t)\,dt &\sim \int_{-\pi}^{x} \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{4}{n} \sin(nt)\,dt \\
F(x) &\sim \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{4}{n} \int_{-\pi}^{x} \sin(nt)\,dt \\
&= \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{4}{n} \left[-\frac{1}{n} \cos(nt) \right]_{-\pi}^{x} \\
&= \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{4}{n^2} (-\cos(nx) + (-1)^n) \\
&= 4 \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{-1}{n^2} - 4 \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{\cos(nx)}{n^2}
\end{aligned}$$

Since this series converges uniformly,

$$4 \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{-1}{n^2} - 4 \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{\cos(nx)}{n^2} = F(x) = \begin{cases} -x - \pi & \textit{for } -\pi \le x < 0 \\ x - \pi & \textit{for } 0 \le x < \pi. \end{cases}$$

The value of the constant term is

$$4 \sum_{\substack{n=1 \\ \text{odd} n}}^{\infty} \frac{-1}{n^2} = \frac{2}{\pi} \int_0^{\pi} F(x)\,\mathrm{d}x = -\frac{1}{\pi}.$$

Thus

$$-\frac{1}{\pi} - 4\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty} \frac{\cos(nx)}{n^2} = \begin{cases} -x-\pi & \textit{for } -\pi \le x < 0 \\ x-\pi & \textit{for } 0 \le x < \pi. \end{cases}$$

Differentiating Fourier Series. Recall that in general, a series can only be differentiated if it is uniformly convergent. The necessary and sufficient condition that a Fourier series be uniformly convergent is that the periodic extension of the function is continuous.

Result 28.10.1 The Fourier series of a function $f(x)$ can be differentiated only if the periodic extension of $f(x)$ is continuous.

Example 28.10.2 *Consider the function defined by*

$$f(x) = \begin{cases} \pi & \textit{for } 0 \le x < \pi \\ -\pi & \textit{for } -\pi \le x < 0. \end{cases}$$

$f(x)$ *has the Fourier series*

$$f(x) \sim \sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty} \frac{4}{n} \sin nx.$$

The function has a derivative except at the points $x = n\pi$. *Differentiating the Fourier series yields*

$$f'(x) \sim 4\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty} \cos(nx).$$

For $x \neq n\pi$, *this implies*

$$0 = 4\sum_{\substack{n=1\\ \text{odd}\, n}}^{\infty} \cos(nx),$$

which is false. The series does not converge. This is as we expected since the Fourier series for $f(x)$ is not uniformly convergent.