Chapter 13

The Residue Theorem

Man will occasionally stumble over the truth, but most of the time he will pick himself up and continue on.

- Winston Churchill

13.1 The Residue Theorem

We will find that many integrals on closed contours may be evaluated in terms of the residues of a function. We first define residues and then prove the Residue Theorem.
**Result 13.1.1 Residues.** Let \( f(z) \) be single-valued an analytic in a deleted neighborhood of \( z_0 \). Then \( f(z) \) has the Laurent series expansion

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,
\]

The residue of \( f(z) \) at \( z = z_0 \) is the coefficient of the \( \frac{1}{z-z_0} \) term:

\[
\text{Res}(f(z), z_0) = a_{-1}.
\]

The residue at a branch point or non-isolated singularity is undefined as the Laurent series does not exist. If \( f(z) \) has a pole of order \( n \) at \( z = z_0 \) then we can use the Residue Formula:

\[
\text{Res}(f(z), z_0) = \lim_{z \to z_0} \left( \frac{1}{(n - 1)!} \frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)] \right).
\]

See Exercise 13.4 for a proof of the Residue Formula.

**Example 13.1.1** In Example 8.4.5 we showed that \( f(z) = z/\sin z \) has first order poles at \( z = n\pi, \ n \in \mathbb{Z} \setminus \{0\} \).
we find the residues at these isolated singularities.

\[
\text{Res} \left( \frac{z}{\sin z}, z = n\pi \right) = \lim_{z \to n\pi} \left( (z - n\pi) \frac{z}{\sin z} \right) \\
= n\pi \lim_{z \to n\pi} \frac{z - n\pi}{\sin z} \\
= n\pi \lim_{z \to n\pi} \frac{1}{\cos z} \\
= n\pi \frac{1}{(-1)^n} \\
= (-1)^n n\pi
\]

Residue Theorem. We can evaluate many integrals in terms of the residues of a function. Suppose \( f(z) \) has only one singularity, (at \( z = z_0 \)), inside the simple, closed, positively oriented contour \( C \). \( f(z) \) has a convergent Laurent series in some deleted disk about \( z_0 \). We deform \( C \) to lie in the disk. See Figure 13.1. We now evaluate \( \int_C f(z) \, dz \) by deforming the contour and using the Laurent series expansion of the function.
\[
\int_C f(z) \, dz = \int_B f(z) \, dz \\
= \int_B \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \, dz \\
= \sum_{n=-\infty}^{\infty} a_n \left[ \frac{(z - z_0)^{n+1}}{n + 1} \right] r e^{i(\theta + 2\pi)} + a_{-1} \left[ \log(z - z_0) \right] r e^{i\theta} \\
= a_{-1} 2\pi
\]

\[
\int_C f(z) \, dz = i2\pi \text{ Res}(f(z), z_0)
\]

Now assume that \( f(z) \) has \( n \) singularities at \( \{z_1, \ldots, z_n\} \). We deform \( C \) to \( n \) contours \( C_1, \ldots, C_n \) which enclose the singularities and lie in deleted disks about the singularities in which \( f(z) \) has convergent Laurent series. See Figure 13.2. We evaluate \( \int_C f(z) \, dz \) by deforming the contour.

\[
\int_C f(z) \, dz = \sum_{k=1}^{n} \int_{C_k} f(z) \, dz = i2\pi \sum_{k=1}^{n} \text{Res}(f(z), z_k)
\]

Now instead let \( f(z) \) be analytic outside and on \( C \) except for isolated singularities at \( \{\zeta_n\} \) in the domain outside \( C \) and perhaps an isolated singularity at infinity. Let \( a \) be any point in the interior of \( C \). To evaluate \( \int_C f(z) \, dz \) we make the change of variables \( \zeta = 1/(z - a) \). This maps the contour \( C \) to \( C' \). (Note that \( C' \) is negatively oriented.) All the points outside \( C \) are mapped to points inside \( C' \) and vice versa. We can then evaluate the integral in terms of the singularities inside \( C' \).
Figure 13.2: Deform the contour $n$ contours which enclose the $n$ singularities.

\[
\oint_C f(z) \, dz = \oint_{C'} f\left(\frac{1}{\zeta} + a\right) \frac{-1}{\zeta^2} \, d\zeta \\
= \oint_{-C'} \frac{1}{z^2} f\left(\frac{1}{z} + a\right) \, dz \\
= i2\pi \sum_{n} \text{Res} \left(\frac{1}{z^2} f\left(\frac{1}{z} + a\right), \frac{1}{\zeta_n - a}\right) + i2\pi \text{Res} \left(\frac{1}{z^2} f\left(\frac{1}{z} + a\right), 0\right).
\]
Figure 13.3: The change of variables $\zeta = 1/(z-a)$.

**Result 13.1.2 Residue Theorem.** If $f(z)$ is analytic in a compact, closed, connected domain $D$ except for isolated singularities at $\{z_n\}$ in the interior of $D$ then

$$
\oint_{\partial D} f(z) \, dz = \sum_k \oint_{C_k} f(z) \, dz = i2\pi \sum_n \text{Res}(f(z), z_n).
$$

Here the set of contours $\{C_k\}$ make up the positively oriented boundary $\partial D$ of the domain $D$. If the boundary of the domain is a single contour $C$ then the formula simplifies.

$$
\oint_C f(z) \, dz = i2\pi \sum_n \text{Res}(f(z), z_n)
$$

If instead $f(z)$ is analytic outside and on $C$ except for isolated singularities at $\{\zeta_n\}$ in the domain outside $C$ and perhaps an isolated singularity at infinity then

$$
\oint_C f(z) \, dz = i2\pi \sum_n \text{Res} \left( \frac{1}{z^2} f \left( \frac{1}{z} + a \right), \frac{1}{\zeta_n - a} \right) + i2\pi \text{Res} \left( \frac{1}{z^2} f \left( \frac{1}{z} + a \right), 0 \right).
$$

Here $a$ is a any point in the interior of $C$. 
Example 13.1.2 Consider

\[
\frac{1}{it2\pi} \int_C \frac{\sin z}{z(z - 1)} \, dz
\]

where \(C\) is the positively oriented circle of radius 2 centered at the origin. Since the integrand is single-valued with only isolated singularities, the Residue Theorem applies. The value of the integral is the sum of the residues from singularities inside the contour.

The only places that the integrand could have singularities are \(z = 0\) and \(z = 1\). Since

\[
\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1,
\]

there is a removable singularity at the point \(z = 0\). There is no residue at this point.

Now we consider the point \(z = 1\). Since \(\sin(z)/z\) is analytic and nonzero at \(z = 1\), that point is a first order pole of the integrand. The residue there is

\[
\text{Res} \left( \frac{\sin z}{z(z - 1)}, z = 1 \right) = \lim_{z \to 1} (z - 1) \frac{\sin z}{z(z - 1)} = \sin(1).
\]

There is only one singular point with a residue inside the path of integration. The residue at this point is \(\sin(1)\). Thus the value of the integral is

\[
\frac{1}{it2\pi} \int_C \frac{\sin z}{z(z - 1)} \, dz = \sin(1)
\]

Example 13.1.3 Evaluate the integral

\[
\int_C \frac{\cot z \coth z}{z^3} \, dz
\]

where \(C\) is the unit circle about the origin in the positive direction.

The integrand is

\[
\frac{\cot z \coth z}{z^3} = \frac{\cos z \cosh z}{z^3 \sin z \sinh z}
\]
\[
\sin z \text{ has zeros at } n\pi. \text{ sinh } z \text{ has zeros at } in\pi. \text{ Thus the only pole inside the contour of integration is at } z = 0. \text{ Since } \sin z \text{ and sinh } z \text{ both have simple zeros at } z = 0,
\]

\[
\sin z = z + \mathcal{O}(z^3), \quad \sinh z = z + \mathcal{O}(z^3)
\]

the integrand has a pole of order 5 at the origin. The residue at \( z = 0 \) is

\[
\lim_{z \to 0} \frac{1}{4!} \frac{d^4}{dz^4} \left( z^5 \cot z \coth z \frac{1}{z^3} \right) = \lim_{z \to 0} \frac{1}{4!} \frac{d^4}{dz^4} \left( z^2 \cot z \coth z \frac{1}{z^3} \right)
\]

\[
= \frac{1}{4!} \lim_{z \to 0} \left( 24 \cot(z) \coth(z) \csc(z)^2 - 32z \cot(z) \csc(z)^4 - 16z \cos(2z) \coth(z) \csc(z)^4 + 22z^2 \cot(z) \coth(z) \csc(z)^4 \right)
\]

\[
+ 2z^2 \cos(3z) \coth(z) \csc(z)^5 + 24 \cot(z) \coth(z) \csc(z)^2 + 24 \csc(z)^2 \csch(z)^2 - 48z \cot(z) \csc(z)^2 \csch(z)^2
\]

\[
- 48z \coth(z) \csc(z)^2 \csch(z)^2 + 24z^2 \cot(z) \coth(z) \csc(z)^2 \csch(z)^2 + 16z^2 \csc(z)^4 \csch(z)^2 + 8z^2 \cos(2z) \csc(z)^4 \csch(z)^2
\]

\[
- 32z \cot(z) \csc(z)^4 - 16z \cosh(2z) \cot(z) \csc(z)^4 + 22z^2 \cot(z) \coth(z) \csc(z)^2 \csch(z)^4 + 16z^2 \csc(z)^2 \csch(z)^4
\]

\[
+ 8z^2 \cosh(2z) \csc(z)^2 \csch(z)^4 + 2z^2 \cosh(3z) \cot(z) \csc(z)^2 \csch(z)^5 \right)
\]

\[
= \frac{1}{4!} \left( -\frac{56}{15} \right)
\]

\[
= -\frac{7}{45}
\]
Since taking the fourth derivative of $z^2 \cot z \coth z$ really sucks, we would like a more elegant way of finding the residue. We expand the functions in the integrand in Taylor series about the origin.

\[
\frac{\cos z \cosh z}{z^3 \sin z \sinh z} = \frac{\left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right) \left(1 + \frac{z^2}{2} + \frac{z^4}{24} + \cdots\right)}{z^3 \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right) \left(z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right)}
\]

\[
= \frac{1 - \frac{z^4}{6} + \cdots}{z^3 \left(z^2 + z^6 \left(\frac{-1}{36} + \frac{1}{60}\right) + \cdots\right)}
\]

\[
= \frac{1}{z^5} \frac{1 - \frac{z^4}{6} + \cdots}{1 - \frac{z^4}{90} + \cdots}
\]

\[
= \frac{1}{z^5} \left(1 - \frac{z^4}{6} + \cdots\right) \left(1 + \frac{z^4}{90} + \cdots\right)
\]

\[
= \frac{1}{z^5} \left(1 - \frac{7}{45} z^4 + \cdots\right)
\]

\[
= \frac{1}{z^5} - \frac{7}{45} \frac{1}{z} + \cdots
\]

Thus we see that the residue is $-\frac{7}{45}$. Now we can evaluate the integral.

\[
\int_C \frac{\cot z \coth z}{z^3} \, dz = -i \frac{14}{45} \pi
\]

### 13.2 Cauchy Principal Value for Real Integrals

#### 13.2.1 The Cauchy Principal Value

First we recap improper integrals. If $f(x)$ has a singularity at $x_0 \in (a \ldots b)$ then

\[
\int_{a}^{b} f(x) \, dx = \lim_{\epsilon \to 0^+} \int_{a}^{x_0 - \epsilon} f(x) \, dx + \lim_{\delta \to 0^+} \int_{x_0 + \delta}^{b} f(x) \, dx.
\]
For integrals on \((-\infty \ldots \infty)\),
\[
\int_{-\infty}^{\infty} f(x) \, dx \equiv \lim_{a \to -\infty, \ b \to \infty} \int_{a}^{b} f(x) \, dx.
\]

**Example 13.2.1** \(\int_{-1}^{1} \frac{1}{x} \, dx \) is divergent. We show this with the definition of improper integrals.

\[
\int_{-1}^{1} \frac{1}{x} \, dx = \lim_{\epsilon \to 0^+} \left( \int_{-\epsilon}^{-1} \frac{1}{x} \, dx + \int_{1}^{1+\epsilon} \frac{1}{x} \, dx \right).
\]

\[
= \lim_{\epsilon \to 0^+} [\ln |x|]_{-\epsilon}^{-1} + \lim_{\delta \to 0^+} [\ln |x|]_{\delta}^{1}
\]

\[
= \lim_{\epsilon \to 0^+} \ln \epsilon - \lim_{\delta \to 0^+} \ln \delta
\]

The integral diverges because \(\epsilon\) and \(\delta\) approach zero independently.

Since \(1/x\) is an odd function, it appears that the area under the curve is zero. Consider what would happen if \(\epsilon\) and \(\delta\) were not independent. If they approached zero symmetrically, \(\delta = \epsilon\), then the value of the integral would be zero.

\[
\lim_{\epsilon \to 0^+} \left( \int_{-\epsilon}^{-1} \frac{1}{x} \, dx + \int_{\epsilon}^{1} \frac{1}{x} \, dx \right) = \lim_{\epsilon \to 0^+} (\ln \epsilon - \ln \epsilon) = 0
\]

We could make the integral have any value we pleased by choosing \(\delta = c\epsilon\). \(^1\)

\[
\lim_{\epsilon \to 0^+} \left( \int_{-\epsilon}^{-1} \frac{1}{x} \, dx + \int_{c\epsilon}^{1} \frac{1}{x} \, dx \right) = \lim_{\epsilon \to 0^+} (\ln \epsilon - \ln (c\epsilon)) = -\ln c
\]

We have seen it is reasonable that
\[
\int_{-1}^{1} \frac{1}{x} \, dx
\]
has some meaning, and if we could evaluate the integral, the most reasonable value would be zero. The **Cauchy principal value** provides us with a way of evaluating such integrals. If \(f(x)\) is continuous on \((a, b)\) except at the point \(x_0 \in (a, b)\)

\(^1\)This may remind you of conditionally convergent series. You can rearrange the terms to make the series sum to any number.
then the Cauchy principal value of the integral is defined

\[ \int_{a}^{b} f(x) \, dx = \lim_{\epsilon \to 0^+} \left( \int_{a}^{x_0-\epsilon} f(x) \, dx + \int_{x_0+\epsilon}^{b} f(x) \, dx \right). \]

The Cauchy principal value is obtained by approaching the singularity symmetrically. The principal value of the integral may exist when the integral diverges. If the integral exists, it is equal to the principal value of the integral.

The Cauchy principal value of \( \int_{-1}^{1} \frac{1}{x} \, dx \) is defined

\[ \int_{-1}^{1} \frac{1}{x} \, dx \equiv \lim_{\epsilon \to 0^+} \left( \int_{-1}^{-\epsilon} \frac{1}{x} \, dx + \int_{\epsilon}^{1} \frac{1}{x} \, dx \right) \]
\[ = \lim_{\epsilon \to 0^+} \left( [\log |x|]_{-\epsilon}^{-1} [\log |x|]_{\epsilon}^{1} \right) \]
\[ = \lim_{\epsilon \to 0^+} (\log | - \epsilon | - \log |\epsilon|) \]
\[ = 0. \]

(Another notation for the principal value of an integral is \( \text{PV} \int f(x) \, dx. \) Since the limits of integration approach zero symmetrically, the two halves of the integral cancel. If the limits of integration approached zero independently, (the definition of the integral), then the two halves would both diverge.

Example 13.2.2 \( \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx \) is divergent. We show this with the definition of improper integrals.

\[ \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{a \to -\infty, b \to \infty} \int_{a}^{b} \frac{x}{x^2 + 1} \, dx \]
\[ = \lim_{a \to -\infty, b \to \infty} \left[ \frac{1}{2} \ln(x^2 + 1) \right]_{a}^{b} \]
\[ = \frac{1}{2} \lim_{a \to -\infty, b \to \infty} \ln \left( \frac{b^2 + 1}{a^2 + 1} \right) \]

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The integral diverges because \( a \) and \( b \) approach infinity independently. Now consider what would happen if \( a \) and \( b \) were not independent. If they approached zero symmetrically, \( a = -b \), then the value of the integral would be zero.

\[
\frac{1}{2} \lim_{b \to \infty} \ln \left( \frac{b^2 + 1}{b^2 + 1} \right) = 0
\]

We could make the integral have any value we pleased by choosing \( a = -cb \).

We can assign a meaning to divergent integrals of the form \( \int_{-\infty}^{\infty} f(x) \, dx \) with the Cauchy principal value. The Cauchy principal value of the integral is defined

\[
\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to \infty} \int_{-a}^{a} f(x) \, dx.
\]

The Cauchy principal value is obtained by approaching infinity symmetrically.

The Cauchy principal value of \( \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx \) is defined

\[
\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{a \to \infty} \int_{-a}^{a} \frac{x}{x^2 + 1} \, dx
\]

\[
= \lim_{a \to \infty} \left[ \frac{1}{2} \ln \left( x^2 + 1 \right) \right]_{-a}^{a}
\]

\[
= 0.
\]
Result 13.2.1 Cauchy Principal Value. If $f(x)$ is continuous on $(a, b)$ except at the point $x_0 \in (a, b)$ then the integral of $f(x)$ is defined

$$
\int_a^b f(x) \, dx = \lim_{\epsilon \to 0^+} \int_a^{x_0-\epsilon} f(x) \, dx + \lim_{\delta \to 0^+} \int_{x_0+\delta}^b f(x) \, dx.
$$

The Cauchy principal value of the integral is defined

$$
\int_a^b f(x) \, dx = \lim_{\epsilon \to 0^+} \left( \int_a^{x_0-\epsilon} f(x) \, dx + \int_{x_0+\epsilon}^b f(x) \, dx \right).
$$

If $f(x)$ is continuous on $(-\infty, \infty)$ then the integral of $f(x)$ is defined

$$
\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty, b \to \infty} \int_a^b f(x) \, dx.
$$

The Cauchy principal value of the integral is defined

$$
\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to \infty} \int_{-a}^{a} f(x) \, dx.
$$

The principal value of the integral may exist when the integral diverges. If the integral exists, it is equal to the principal value of the integral.

Example 13.2.3 Clearly $\int_{-\infty}^{\infty} x \, dx$ diverges, however the Cauchy principal value exists.

$$
\int_{-\infty}^{\infty} x \, dx = \lim_{a \to \infty} \left[ \frac{x^2}{2} \right]_{-a}^a = 0
$$
In general, if \( f(x) \) is an odd function with no singularities on the finite real axis then

\[
\int_{-\infty}^{\infty} f(x) \, dx = 0.
\]

### 13.3 Cauchy Principal Value for Contour Integrals

**Example 13.3.1** Consider the integral

\[
\int_{C_r} \frac{1}{z-1} \, dz,
\]

where \( C_r \) is the positively oriented circle of radius \( r \) and center at the origin. From the residue theorem, we know that the integral is

\[
\int_{C_r} \frac{1}{z-1} \, dz = \begin{cases} 0 & \text{for } r < 1, \\ i2\pi & \text{for } r > 1. \end{cases}
\]

When \( r = 1 \), the integral diverges, as there is a first order pole on the path of integration. However, the principal value of the integral exists.

\[
\int_{C_r} \frac{1}{z-1} \, dz = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{2\pi-\epsilon} \frac{1}{e^{i\theta} - 1} i e^{i\theta} \, d\theta
\]

\[= \lim_{\epsilon \to 0^+} \left[ \log(e^{i\theta} - 1) \right]_{\epsilon}^{2\pi-\epsilon} \]
We choose the branch of the logarithm with a branch cut on the positive real axis and \( \arg \log z \in (0, 2\pi) \).

\[
\begin{align*}
&= \lim_{\epsilon \to 0^+} \left( \log \left( e^{i(2\pi - \epsilon)} - 1 \right) - \log \left( e^{i\epsilon} - 1 \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left( \log \left( (1 - i\epsilon + O(\epsilon^2)) - 1 \right) - \log \left( (1 + i\epsilon + O(\epsilon^2)) - 1 \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left( \log \left( -i\epsilon + O(\epsilon^2) \right) - \log \left( i\epsilon + O(\epsilon^2) \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left( \log \left( (\epsilon + O(\epsilon^2)) + i \arg (-i\epsilon + O(\epsilon^2)) \right) - \log \left( (\epsilon + O(\epsilon^2)) - i \arg (i\epsilon + O(\epsilon^2)) \right) \right) \\
&= \lim_{\epsilon \to 0^+} \left( \log \left( -i\epsilon + O(\epsilon^2) \right) - \log \left( i\epsilon + O(\epsilon^2) \right) \right) \\
&= i \frac{3\pi}{2} - i \frac{\pi}{2} \\
&= i\pi
\end{align*}
\]

Thus we obtain

\[
\int_{C_r} \frac{1}{z-1} \, dz = \begin{cases} 
0 & \text{for } r < 1, \\
 i\pi & \text{for } r = 1, \\
 i2\pi & \text{for } r > 1.
\end{cases}
\]

In the above example we evaluated the contour integral by parameterizing the contour. This approach is only feasible when the integrand is simple. We would like to use the residue theorem to more easily evaluate the principal value of the integral. But before we do that, we will need a preliminary result.

**Result 13.3.1** Let \( f(z) \) have a first order pole at \( z = z_0 \) and let \((z - z_0)f(z)\) be analytic in some neighborhood of \( z_0 \). Let the contour \( C_\epsilon \) be a circular arc from \( z_0 + \epsilon e^{i\alpha} \) to \( z_0 + \epsilon e^{i\beta} \). (We assume that \( \beta > \alpha \) and \( \beta - \alpha < 2\pi \).)

\[
\lim_{\epsilon \to 0^+} \int_{C_\epsilon} f(z) \, dz = i(\beta - \alpha) \text{Res}(f(z), z_0)
\]

The contour is shown in Figure 13.4. (See Exercise 13.9 for a proof of this result.)
Example 13.3.2 Consider

\[ \int_C \frac{1}{z-1} \, dz \]

where \( C \) is the unit circle. Let \( C_p \) be the circular arc of radius 1 that starts and ends a distance of \( \epsilon \) from \( z = 1 \). Let \( C_\epsilon \) be the positive, circular arc of radius \( \epsilon \) with center at \( z = 1 \) that joins the endpoints of \( C_p \). Let \( C_i \), be the union of \( C_p \) and \( C_\epsilon \). (\( C_p \) stands for Principal value Contour; \( C_i \) stands for Indented Contour.) \( C_i \) is an indented contour that avoids the first order pole at \( z = 1 \). Figure 13.5 shows the three contours.
Note that the principal value of the integral is

\[ \int_C \frac{1}{z - 1} \, dz = \lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{1}{z - 1} \, dz. \]

We can calculate the integral along \( C_i \) with the residue theorem.

\[ \int_{C_i} \frac{1}{z - 1} \, dz = i2\pi \]

We can calculate the integral along \( C_\epsilon \) using Result 13.3.1. Note that as \( \epsilon \to 0^+ \), the contour becomes a semi-circle, a circular arc of \( \pi \) radians.

\[ \lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{1}{z - 1} \, dz = i\pi \text{ Res} \left( \frac{1}{z - 1}, 1 \right) = i\pi \]

Now we can write the principal value of the integral along \( C \) in terms of the two known integrals.

\[ \int_C \frac{1}{z - 1} \, dz = \int_{C_i} \frac{1}{z - 1} \, dz - \int_{C_\epsilon} \frac{1}{z - 1} \, dz \]
\[ = i2\pi - i\pi \]
\[ = i\pi \]

In the previous example, we formed an indented contour that included the first order pole. You can show that if we had indented the contour to exclude the pole, we would obtain the same result. (See Exercise 13.11.)

We can extend the residue theorem to principal values of integrals. (See Exercise 13.10.)
Result 13.3.2 Residue Theorem for Principal Values. Let \( f(z) \) be analytic inside and on a simple, closed, positive contour \( C \), except for isolated singularities at \( z_1, \ldots, z_m \) inside the contour and first order poles at \( \zeta_1, \ldots, \zeta_n \) on the contour. Further, let the contour be \( C^1 \) at the locations of these first order poles. (i.e., the contour does not have a corner at any of the first order poles.) Then the principal value of the integral of \( f(z) \) along \( C \) is

\[
\int_C f(z) \, dz = i2\pi \sum_{j=1}^{m} \text{Res}(f(z), z_j) + i\pi \sum_{j=1}^{n} \text{Res}(f(z), \zeta_j).
\]

13.4 Integrals on the Real Axis

Example 13.4.1 We wish to evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx.
\]

We can evaluate this integral directly using calculus.

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx = \arctan x \bigg|_{-\infty}^{\infty} = \pi
\]

Now we will evaluate the integral using contour integration. Let \( C_R \) be the semicircular arc from \( R \) to \( -R \) in the upper half plane. Let \( C \) be the union of \( C_R \) and the interval \([-R, R]\).

We can evaluate the integral along \( C \) with the residue theorem. The integrand has first order poles at \( z = \pm i \). For
$R > 1$, we have

$$\int_{C} \frac{1}{z^2 + 1} \, dz = i2\pi \text{Res} \left( \frac{1}{z^2 + 1}, i \right)$$

$$= i2\pi \frac{1}{i2}$$

$$= \pi.$$

Now we examine the integral along $C_R$. We use the maximum modulus integral bound to show that the value of the integral vanishes as $R \to \infty$.

$$\left| \int_{C_R} \frac{1}{z^2 + 1} \, dz \right| \leq \pi R \max_{z \in C_R} \left| \frac{1}{z^2 + 1} \right|$$

$$= \pi R \frac{1}{R^2 - 1}$$

$$\to 0 \quad \text{as} \quad R \to \infty.$$

Now we are prepared to evaluate the original real integral.

$$\int_{C} \frac{1}{z^2 + 1} \, dz = \pi$$

$$\int_{-R}^{R} \frac{1}{x^2 + 1} \, dx + \int_{CR} \frac{1}{z^2 + 1} \, dz = \pi$$

We take the limit as $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx = \pi$$

We would get the same result by closing the path of integration in the lower half plane. Note that in this case the closed contour would be in the negative direction.
If you are really observant, you may have noticed that we did something a little funny in evaluating
\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx. \]

The definition of this improper integral is
\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{a \to +\infty} \int_{-a}^{0} \frac{1}{x^2 + 1} \, dx + \lim_{b \to +\infty} \int_{0}^{b} \frac{1}{x^2 + 1} \, dx. \]

In the above example we instead computed
\[ \lim_{R \to +\infty} \int_{-R}^{R} \frac{1}{x^2 + 1} \, dx. \]

Note that for some integrands, the former and latter are not the same. Consider the integral of \( \frac{x}{x^2 + 1} \).
\[ \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{a \to +\infty} \int_{-a}^{0} \frac{x}{x^2 + 1} \, dx + \lim_{b \to +\infty} \int_{0}^{b} \frac{x}{x^2 + 1} \, dx \]
\[ = \lim_{a \to +\infty} \left( \frac{1}{2} \log \left| a^2 + 1 \right| \right) + \lim_{b \to +\infty} \left( -\frac{1}{2} \log \left| b^2 + 1 \right| \right) \]

Note that the limits do not exist and hence the integral diverges. We get a different result if the limits of integration approach infinity symmetrically.
\[ \lim_{R \to +\infty} \int_{-R}^{R} \frac{x}{x^2 + 1} \, dx = \lim_{R \to +\infty} \left( \frac{1}{2} \left( \log \left| R^2 + 1 \right| - \log \left| R^2 + 1 \right| \right) \right) \]
\[ = 0 \]
(Note that the integrand is an odd function, so the integral from \(-R\) to \(R\) is zero.) We call this the principal value of the integral and denote it by writing “PV” in front of the integral sign or putting a dash through the integral.

\[ \text{PV} \int_{-\infty}^{\infty} f(x) \, dx \equiv \int_{-\infty}^{\infty} f(x) \, dx \equiv \lim_{R \to +\infty} \int_{-R}^{R} f(x) \, dx \]
The principal value of an integral may exist when the integral diverges. If the integral does converge, then it is equal to its principal value.

We can use the method of Example 13.4.1 to evaluate the principal value of integrals of functions that vanish fast enough at infinity.

**Result 13.4.1** Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Let $C_R$ be the semi-circle from $R$ to $-R$ in the upper half plane. If

$$\lim_{R \to \infty} \left( R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$\int_{-\infty}^{\infty} f(x) \, dx = i 2 \pi \sum_{k=1}^{m} \text{Res} \left( f(z), z_k \right) + i \pi \sum_{k=1}^{n} \text{Res} \left( f(z), x_k \right)$$

where $z_1, \ldots z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis.

Now let $C_R$ be the semi-circle from $R$ to $-R$ in the lower half plane. If

$$\lim_{R \to \infty} \left( R \max_{z \in C_R} |f(z)| \right) = 0$$

then

$$\int_{-\infty}^{\infty} f(x) \, dx = -i 2 \pi \sum_{k=1}^{m} \text{Res} \left( f(z), z_k \right) - i \pi \sum_{k=1}^{n} \text{Res} \left( f(z), x_k \right)$$

where $z_1, \ldots z_m$ are the singularities of $f(z)$ in the lower half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis.
This result is proved in Exercise 13.13. Of course we can use this result to evaluate the integrals of the form

\[ \int_{0}^{\infty} f(z) \, dz, \]

where \( f(x) \) is an even function.

### 13.5 Fourier Integrals

In order to do Fourier transforms, which are useful in solving differential equations, it is necessary to be able to calculate Fourier integrals. Fourier integrals have the form

\[ \int_{-\infty}^{\infty} e^{i\omega x} f(x) \, dx. \]

We evaluate these integrals by closing the path of integration in the lower or upper half plane and using techniques of contour integration.

Consider the integral

\[ \int_{0}^{\pi/2} e^{-R\sin \theta} \, d\theta. \]

Since \( 2\theta/\pi \leq \sin \theta \) for \( 0 \leq \theta \leq \pi/2 \),

\[ e^{-R\sin \theta} \leq e^{-R\theta/\pi} \quad \text{for} \quad 0 \leq \theta \leq \pi/2 \]

\[ \int_{0}^{\pi/2} e^{-R\sin \theta} \, d\theta \leq \int_{0}^{\pi/2} e^{-R\theta/\pi} \, d\theta \]

\[ = \left[ -\frac{\pi}{2R} e^{-R\theta/\pi} \right]^{\pi/2}_{0} \]

\[ = -\frac{\pi}{2R} (e^{-R} - 1) \]

\[ \leq \frac{\pi}{2R} \]

\[ \to 0 \quad \text{as} \quad R \to \infty \]
We can use this to prove the following Result 13.5.1. (See Exercise 13.17.)

**Result 13.5.1 Jordan’s Lemma.**

\[ \int_0^\pi e^{-R\sin \theta} \, d\theta < \frac{\pi}{R}. \]

Suppose that \( f(z) \) vanishes as \( |z| \to \infty \). If \( \omega \) is a (positive/negative) real number and \( C_R \) is a semi-circle of radius \( R \) in the (upper/lower) half plane then the integral

\[ \int_{C_R} f(z) e^{i\omega z} \, dz \]

vanishes as \( R \to \infty \).

We can use Jordan’s Lemma and the Residue Theorem to evaluate many Fourier integrals. Consider \( \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx \), where \( \omega \) is a positive real number. Let \( f(z) \) be analytic except for isolated singularities, with only first order poles on the real axis. Let \( C \) be the contour from \( -R \) to \( R \) on the real axis and then back to \( -R \) along a semi-circle in the upper half plane. If \( R \) is large enough so that \( C \) encloses all the singularities of \( f(z) \) in the upper half plane then

\[ \int_C f(z) e^{i\omega z} \, dz = i2\pi \sum_{k=1}^{m} \text{Res}(f(z) e^{i\omega z}, z_k) + i\pi \sum_{k=1}^{n} \text{Res}(f(z) e^{i\omega z}, x_k) \]

where \( z_1, \ldots, z_m \) are the singularities of \( f(z) \) in the upper half plane and \( x_1, \ldots, x_n \) are the first order poles on the real axis. If \( f(z) \) vanishes as \( |z| \to \infty \) then the integral on \( C_R \) vanishes as \( R \to \infty \) by Jordan’s Lemma.

\[ \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx = i2\pi \sum_{k=1}^{m} \text{Res}(f(z) e^{i\omega z}, z_k) + i\pi \sum_{k=1}^{n} \text{Res}(f(z) e^{i\omega z}, x_k) \]

For negative \( \omega \) we close the path of integration in the lower half plane. Note that the contour is then in the negative direction.
**Result 13.5.2 Fourier Integrals.** Let \( f(z) \) be analytic except for isolated singularities, with only first order poles on the real axis. Suppose that \( f(z) \) vanishes as \( |z| \to \infty \). If \( \omega \) is a positive real number then

\[
\int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx = i2\pi \sum_{k=1}^{m} \text{Res}(f(z) e^{i\omega z}, z_k) + i\pi \sum_{k=1}^{n} \text{Res}(f(z) e^{i\omega z}, x_k)
\]

where \( z_1, \ldots, z_m \) are the singularities of \( f(z) \) in the upper half plane and \( x_1, \ldots, x_n \) are the first order poles on the real axis. If \( \omega \) is a negative real number then

\[
\int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx = -i2\pi \sum_{k=1}^{m} \text{Res}(f(z) e^{i\omega z}, z_k) - i\pi \sum_{k=1}^{n} \text{Res}(f(z) e^{i\omega z}, x_k)
\]

where \( z_1, \ldots, z_m \) are the singularities of \( f(z) \) in the lower half plane and \( x_1, \ldots, x_n \) are the first order poles on the real axis.

### 13.6 Fourier Cosine and Sine Integrals

Fourier cosine and sine integrals have the form,

\[
\int_{0}^{\infty} f(x) \cos(\omega x) \, dx \quad \text{and} \quad \int_{0}^{\infty} f(x) \sin(\omega x) \, dx.
\]

If \( f(x) \) is even/odd then we can evaluate the cosine/sine integral with the method we developed for Fourier integrals.
Let \( f(z) \) be analytic except for isolated singularities, with only first order poles on the real axis. Suppose that \( f(x) \) is an even function and that \( f(z) \) vanishes as \( |z| \to \infty \). We consider real \( \omega > 0 \).

\[
\int_{0}^{\infty} f(x) \cos(\omega x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx
\]

Since \( f(x) \sin(\omega x) \) is an odd function,

\[
\frac{1}{2} \int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx = 0.
\]

Thus

\[
\int_{0}^{\infty} f(x) \cos(\omega x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx
\]

Now we apply Result 13.5.2.

\[
\int_{0}^{\infty} f(x) \cos(\omega x) \, dx = \frac{i\pi}{2} \sum_{k=1}^{m} \text{Res}(f(z) e^{i\omega z}, z_k) + \frac{i\pi}{2} \sum_{k=1}^{n} \text{Res}(f(z) e^{i\omega z}, x_k)
\]

where \( z_1, \ldots, z_m \) are the singularities of \( f(z) \) in the upper half plane and \( x_1, \ldots, x_n \) are the first order poles on the real axis.

If \( f(x) \) is an odd function, we note that \( f(x) \cos(\omega x) \) is an odd function to obtain the analogous result for Fourier sine integrals.
Result 13.6.1 Fourier Cosine and Sine Integrals. Let $f(z)$ be analytic except for isolated singularities, with only first order poles on the real axis. Suppose that $f(x)$ is an even function and that $f(z)$ vanishes as $|z| \to \infty$. We consider real $\omega > 0$.

$$\int_{0}^{\infty} f(x) \cos(\omega x) \, dx = \nu \pi \sum_{k=1}^{m} \text{Res}(f(z) \, e^{\omega z}, z_k) + \frac{\nu \pi}{2} \sum_{k=1}^{n} \text{Res}(f(z) \, e^{\omega z}, x_k)$$

where $z_1, \ldots, z_m$ are the singularities of $f(z)$ in the upper half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis. If $f(x)$ is an odd function then,

$$\int_{0}^{\infty} f(x) \sin(\omega x) \, dx = \pi \sum_{k=1}^{\mu} \text{Res}(f(z) \, e^{\omega z}, \zeta_k) + \frac{\pi}{2} \sum_{k=1}^{n} \text{Res}(f(z) \, e^{\omega z}, x_k)$$

where $\zeta_1, \ldots, \zeta_\mu$ are the singularities of $f(z)$ in the lower half plane and $x_1, \ldots, x_n$ are the first order poles on the real axis.

Now suppose that $f(x)$ is neither even nor odd. We can evaluate integrals of the form:

$$\int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx$$

by writing them in terms of Fourier integrals

$$\int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \, e^{\omega x} \, dx + \frac{1}{2} \int_{-\infty}^{\infty} f(x) \, e^{-\omega x} \, dx$$

$$\int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx = -\frac{i}{2} \int_{-\infty}^{\infty} f(x) \, e^{\omega x} \, dx + \frac{i}{2} \int_{-\infty}^{\infty} f(x) \, e^{-\omega x} \, dx$$
13.7 Contour Integration and Branch Cuts

Example 13.7.1 Consider

\[ \int_0^\infty \frac{x^{-a}}{x+1} \, dx, \quad 0 < a < 1, \]

where \( x^{-a} \) denotes \( \exp(-a \ln(x)) \). We choose the branch of the function

\[ f(z) = \frac{z^{-a}}{z+1} \quad |z| > 0, \quad 0 < \arg z < 2\pi \]

with a branch cut on the positive real axis.

Let \( C_\epsilon \) and \( C_R \) denote the circular arcs of radius \( \epsilon \) and \( R \) where \( \epsilon < 1 < R \). \( C_\epsilon \) is negatively oriented; \( C_R \) is positively oriented. Consider the closed contour \( C \) that is traced by a point moving from \( C_\epsilon \) to \( C_R \) above the branch cut, next around \( C_R \), then below the cut to \( C_\epsilon \), and finally around \( C_\epsilon \). (See Figure 13.6.)

We write \( f(z) \) in polar coordinates.

\[ f(z) = \frac{\exp(-a \log z)}{z+1} = \frac{\exp(-a(\log r + i\theta))}{r e^{i\theta} + 1} \]
We evaluate the function above, \((z = r e^{i0})\), and below, \((z = r e^{i2\pi})\), the branch cut.

\[
\begin{align*}
  f(r e^{i0}) &= \frac{\exp[-a(\log r + i0)]}{r + 1} = \frac{r^{-a}}{r + 1} \\
  f(r e^{i2\pi}) &= \frac{\exp[-a(\log r + i2\pi)]}{r + 1} = \frac{r^{-a} e^{-i2a\pi}}{r + 1}.
\end{align*}
\]

We use the residue theorem to evaluate the integral along \(C\).

\[
\oint_C f(z) \, dz = i2\pi \text{ Res}(f(z), -1)
\]

\[
\int_{\epsilon}^{R} \frac{r^{-a}}{r + 1} \, dr + \int_{C_R} f(z) \, dz - \int_{\epsilon}^{R} \frac{r^{-a} e^{-i2a\pi}}{r + 1} \, dr + \int_{C_\epsilon} f(z) \, dz = i2\pi \text{ Res}(f(z), -1)
\]

The residue is

\[
\text{Res}(f(z), -1) = \exp(-a \log(-1)) = \exp(-a(\log 1 + i\pi)) = e^{-ia\pi}.
\]

We bound the integrals along \(C_\epsilon\) and \(C_R\) with the maximum modulus integral bound.

\[
\left| \int_{C_\epsilon} f(z) \, dz \right| \leq 2\pi \epsilon \frac{\epsilon^{-a}}{1 - \epsilon} = 2\pi \frac{\epsilon^{1-a}}{1 - \epsilon}
\]

\[
\left| \int_{C_R} f(z) \, dz \right| \leq 2\pi R \frac{R^{-a}}{R - 1} = 2\pi \frac{R^{1-a}}{R - 1}
\]

Since \(0 < a < 1\), the values of the integrals tend to zero as \(\epsilon \to 0\) and \(R \to \infty\). Thus we have

\[
\int_{0}^{\infty} \frac{r^{-a}}{r + 1} \, dr = i2\pi \frac{e^{-ia\pi}}{1 - e^{-i2a\pi}}
\]

\[
\int_{0}^{\infty} \frac{x^{-a}}{x + 1} \, dx = \frac{\pi}{\sin a\pi}
\]

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Result 13.7.1 Integrals from Zero to Infinity. Let \( f(z) \) be a single-valued analytic function with only isolated singularities and no singularities on the positive, real axis, \([0, \infty)\). Let \( a \notin \mathbb{Z} \). If the integrals exist then,

\[
\int_0^\infty f(x) \, dx = -\sum_{k=1}^n \text{Res} \left( f(z) \log z, z_k \right),
\]

\[
\int_0^\infty x^a f(x) \, dx = \frac{i2\pi}{1 - e^{i2\pi a}} \sum_{k=1}^n \text{Res} \left( z^a f(z), z_k \right),
\]

\[
\int_0^\infty f(x) \log x \, dx = -\frac{1}{2} \sum_{k=1}^n \text{Res} \left( f(z) \log^2 z, z_k \right) + i\pi \sum_{k=1}^n \text{Res} \left( f(z) \log z, z_k \right),
\]

\[
\int_0^\infty x^a f(x) \log x \, dx = \frac{i2\pi}{1 - e^{i2\pi a}} \sum_{k=1}^n \text{Res} \left( z^a f(z) \log z, z_k \right)
\]

\[
+ \frac{\pi^2 a}{\sin^2(\pi a)} \sum_{k=1}^n \text{Res} \left( z^a f(z), z_k \right),
\]

\[
\int_0^\infty x^a f(x) \log^m x \, dx = \frac{\partial^m}{\partial a^m} \left( \frac{i2\pi}{1 - e^{i2\pi a}} \sum_{k=1}^n \text{Res} \left( z^a f(z), z_k \right) \right),
\]

where \( z_1, \ldots, z_n \) are the singularities of \( f(z) \) and there is a branch cut on the positive real axis with \( 0 < \arg(z) < 2\pi \).
13.8 Exploiting Symmetry

We have already used symmetry of the integrand to evaluate certain integrals. For $f(x)$ an even function we were able to evaluate $\int_{0}^{\infty} f(x) \, dx$ by extending the range of integration from $-\infty$ to $\infty$. For

$$\int_{0}^{\infty} x^\alpha f(x) \, dx$$

we put a branch cut on the positive real axis and noted that the value of the integrand below the branch cut is a constant multiple of the value of the function above the branch cut. This enabled us to evaluate the real integral with contour integration. In this section we will use other kinds of symmetry to evaluate integrals. We will discover that periodicity of the integrand will produce this symmetry.

13.8.1 Wedge Contours

We note that $z^n = r^n e^{i n \theta}$ is periodic in $\theta$ with period $2\pi / n$. The real and imaginary parts of $z^n$ are odd periodic in $\theta$ with period $\pi / n$. This observation suggests that certain integrals on the positive real axis may be evaluated by closing the path of integration with a wedge contour.

Example 13.8.1 Consider

$$\int_{0}^{\infty} \frac{1}{1 + x^n} \, dx$$
where \( n \in \mathbb{N}, n \geq 2 \). We can evaluate this integral using Result 13.7.1.

\[
\int_0^\infty \frac{1}{1 + x^n} \, dx = - \sum_{k=0}^{n-1} \text{Res} \left( \frac{\log z}{1 + z^n} e^{i\pi(1+2k)/n} \right)
\]

\[
= - \sum_{k=0}^{n-1} \lim_{z \to e^{i\pi(1+2k)/n}} \left( \frac{z - e^{i\pi(1+2k)/n} \log z}{1 + z^n} \right)
\]

\[
= - \sum_{k=0}^{n-1} \lim_{z \to e^{i\pi(1+2k)/n}} \left( \frac{\log z + (z - e^{i\pi(1+2k)/n})/z}{n z^{n-1}} \right)
\]

\[
= - \sum_{k=0}^{n-1} \left( \frac{i\pi (1 + 2k)/n}{n e^{i\pi(1+2k)(n-1)/n}} \right)
\]

\[
= - \frac{i\pi}{n^2 e^{i\pi(n-1)/n}} \sum_{k=0}^{n-1} (1 + 2k) e^{i2\pi k/n}
\]

\[
= \frac{i2\pi e^{i\pi/n}}{n^2} \sum_{k=1}^{n-1} k e^{i2\pi k/n}
\]

\[
= \frac{i2\pi e^{i\pi/n}}{n^2} \frac{n}{e^{i2\pi/n} - 1}
\]

\[
= \frac{i2\pi e^{i\pi/n}}{n^2} \frac{\pi}{e^{i2\pi/n} - 1}
\]

\[
= \frac{\pi}{n \sin(\pi/n)}
\]

This is a bit grungy. To find a spiffier way to evaluate the integral we note that if we write the integrand as a function of \( r \) and \( \theta \), it is periodic in \( \theta \) with period \( 2\pi/n \).

\[
\frac{1}{1 + z^n} = \frac{1}{1 + r^n e^{i n \theta}}
\]

The integrand along the rays \( \theta = 2\pi/n, 4\pi/n, 6\pi/n, \ldots \) has the same value as the integrand on the real axis. Consider the contour \( C \) that is the boundary of the wedge \( 0 < r < R, 0 < \theta < 2\pi/n \). There is one singularity inside the
We evaluate the residue there.

\[
\begin{align*}
\text{Res} \left( \frac{1}{1 + z^n}, e^{i\pi/n} \right) &= \lim_{z \to e^{i\pi/n}} \frac{z - e^{i\pi/n}}{1 + z^n} \\
&= \lim_{z \to e^{i\pi/n}} \frac{1}{nz^{n-1}} \\
&= \frac{e^{i\pi/n}}{n}
\end{align*}
\]

We evaluate the integral along \( C \) with the residue theorem.

\[
\int_C \frac{1}{1 + z^n} \, dz = \frac{-i2\pi e^{i\pi/n}}{n}
\]

Let \( C_R \) be the circular arc. The integral along \( C_R \) vanishes as \( R \to \infty \).

\[
\left| \int_{C_R} \frac{1}{1 + z^n} \, dz \right| \leq \frac{2\pi R}{n} \max_{z \in C_R} \left| \frac{1}{1 + z^n} \right| \\
\leq \frac{2\pi R}{n} \frac{1}{R^n - 1} \\
\to 0 \text{ as } R \to \infty
\]

We parametrize the contour to evaluate the desired integral.

\[
\begin{align*}
\int_0^\infty \frac{1}{1 + x^n} \, dx + \int_0^0 \frac{1}{1 + x^n} e^{i2\pi/n} \, dx &= \frac{-i2\pi e^{i\pi/n}}{n} \\
\int_0^\infty \frac{1}{1 + x^n} \, dx &= \frac{-i2\pi e^{i\pi/n}}{n(1 - e^{i2\pi/n})} \\
\int_0^\infty \frac{1}{1 + x^n} \, dx &= \frac{\pi}{n \sin(\pi/n)}
\end{align*}
\]
13.8.2 Box Contours

Recall that \( e^z = e^{x+iy} \) is periodic in \( y \) with period \( 2\pi \). This implies that the hyperbolic trigonometric functions \( \cosh z, \sinh z \) and \( \tanh z \) are periodic in \( y \) with period \( 2\pi \) and odd periodic in \( y \) with period \( \pi \). We can exploit this property to evaluate certain integrals on the real axis by closing the path of integration with a box contour.

Example 13.8.2 Consider the integral

\[
\int_{-\infty}^{\infty} \frac{1}{\cosh x} \, dx = \left[ i \log \left( \tanh \left( \frac{i\pi}{4} + \frac{x}{2} \right) \right) \right]_{-\infty}^{\infty} \\
= i \log(1) - i \log(-1) \\
= \pi.
\]

We will evaluate this integral using contour integration. Note that

\[
cosh(x + i\pi) = \frac{e^{x+i\pi} + e^{-x-i\pi}}{2} = -\cosh(x).
\]

Consider the box contour \( C \) that is the boundary of the region \(-R < x < R, 0 < y < \pi\). The only singularity of the integrand inside the contour is a first order pole at \( z = i\pi/2 \). We evaluate the integral along \( C \) with the residue theorem.

\[
\oint_C \frac{1}{\cosh z} \, dz = i2\pi \operatorname{Res} \left( \frac{1}{\cosh z}, \frac{i\pi}{2} \right) \\
= i2\pi \lim_{z \to i\pi/2} \frac{z - i\pi/2}{\cosh z} \\
= i2\pi \lim_{z \to i\pi/2} \frac{1}{\sinh z} \\
= 2\pi
\]
The integrals along the sides of the box vanish as \( R \to \infty \).

\[
\left| \int_{\pm R}^{\pm R + i\pi} \frac{1}{\cosh z} \, dz \right| \leq \pi \max_{z \in [\pm R \ldots \pm R + i\pi]} \left| \frac{1}{\cosh z} \right| \\
\leq \pi \max_{y \in [0 \ldots \pi]} \left| \frac{2}{e^{R + iy} + e^{R - iy}} \right| \\
= \frac{2}{e^R - e^{-R}} \\
\leq \frac{\pi}{\sinh R} \\
\to 0 \text{ as } R \to \infty
\]

The value of the integrand on the top of the box is the negative of its value on the bottom. We take the limit as \( R \to \infty \).

\[
\int_{-\infty}^{\infty} \frac{1}{\cosh x} \, dx + \int_{-\infty}^{\infty} \frac{1}{-\cosh x} \, dx = 2\pi \\
\int_{-\infty}^{\infty} \frac{1}{\cosh x} \, dx = \pi
\]

### 13.9 Definite Integrals Involving Sine and Cosine

#### Example 13.9.1
For real-valued \( a \), evaluate the integral:

\[
f(a) = \int_{0}^{2\pi} \frac{d\theta}{1 + a \sin \theta}.
\]

What is the value of the integral for complex-valued \( a \).

**Real-Valued** \( a \). For \(-1 < a < 1\), the integrand is bounded, hence the integral exists. For \(|a| = 1\), the integrand has a second order pole on the path of integration. For \(|a| > 1\) the integrand has two first order poles on the path of integration. The integral is divergent for these two cases. Thus we see that the integral exists for \(-1 < a < 1\).
For \( a = 0 \), the value of the integral is \( 2\pi \). Now consider \( a \neq 0 \). We make the change of variables \( z = e^{i\theta} \). The real integral from \( \theta = 0 \) to \( \theta = 2\pi \) becomes a contour integral along the unit circle, \(|z| = 1\). We write the sine, cosine and the differential in terms of \( z \).

\[
\sin \theta = \frac{z - z^{-1}}{i2}, \quad \cos \theta = \frac{z + z^{-1}}{2}, \quad dz = ie^{i\theta} \, d\theta, \quad d\theta = \frac{dz}{iz}
\]

We write \( f(a) \) as an integral along \( C \), the positively oriented unit circle \(|z| = 1\).

\[
f(a) = \oint_C \frac{1/(iz)}{1 + a(z - z^{-1})/(2i)} \, dz = \oint_C \frac{2/a}{z^2 + (i2/a)z - 1} \, dz
\]

We factor the denominator of the integrand.

\[
f(a) = \oint_C \frac{2/a}{(z - z_1)(z - z_2)} \, dz
\]

\[
z_1 = i\left(\frac{-1 + \sqrt{1 - a^2}}{a}\right), \quad z_2 = i\left(\frac{-1 - \sqrt{1 - a^2}}{a}\right)
\]

Because \(|a| < 1\), the second root is outside the unit circle.

\[
|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1.
\]

Since \(|z_1 z_2| = 1\), \(|z_1| < 1\). Thus the pole at \( z_1 \) is inside the contour and the pole at \( z_2 \) is outside. We evaluate the contour integral with the residue theorem.

\[
f(a) = \oint_C \frac{2/a}{z^2 + (i2/a)z - 1} \, dz
\]

\[
= i2\pi \frac{2/a}{z_1 - z_2}
\]

\[
= i2\pi \frac{1}{i\sqrt{1 - a^2}}
\]

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Complex-Valued $a$. We note that the integral converges except for real-valued $a$ satisfying $|a| \geq 1$. On any closed subset of $\mathbb{C} \setminus \{a \in \mathbb{R} \mid |a| \geq 1\}$ the integral is uniformly convergent. Thus except for the values $\{a \in \mathbb{R} \mid |a| \geq 1\}$, we can differentiate the integral with respect to $a$. $f(a)$ is analytic in the complex plane except for the set of points on the real axis: $a \in (\infty \ldots -1]$ and $a \in [1 \ldots \infty)$. The value of the analytic function $f(a)$ on the real axis for the interval $(-1 \ldots 1)$ is

$$f(a) = \frac{2\pi}{\sqrt{1-a^2}}.$$

By analytic continuation we see that the value of $f(a)$ in the complex plane is the branch of the function

$$f(a) = \frac{2\pi}{(1-a^2)^{1/2}}$$

where $f(a)$ is positive, real-valued for $a \in (-1 \ldots 1)$ and there are branch cuts on the real axis on the intervals: $(-\infty \ldots -1]$ and $[1 \ldots \infty)$.

**Result 13.9.1** For evaluating integrals of the form

$$\int_{\alpha}^{\alpha+2\pi} F(\sin \theta, \cos \theta) \, d\theta$$

it may be useful to make the change of variables $z = e^{i\theta}$. This gives us a contour integral along the unit circle about the origin. We can write the sine, cosine and differential in terms of $z$.

$$\sin \theta = \frac{z - z^{-1}}{i2}, \quad \cos \theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{iz}.$$
13.10 Infinite Sums

The function \( g(z) = \pi \cot(\pi z) \) has simple poles at \( z = n \in \mathbb{Z} \). The residues at these points are all unity.

\[
\text{Res}(\pi \cot(\pi z), n) = \lim_{z \to n} \frac{\pi (z - n) \cos(\pi z)}{\sin(\pi z)}
= \lim_{z \to n} \frac{\pi \cos(\pi z) - \pi (z - n) \sin(\pi z)}{\pi \cos(\pi z)}
= 1
\]

Let \( C_n \) be the square contour with corners at \( z = (n + 1/2)(\pm 1 \pm i) \). Recall that

\[
\cos z = \cos x \cosh y - i \sin x \sinh y \quad \text{and} \quad \sin z = \sin x \cosh y + i \cos x \sinh y.
\]

First we bound the modulus of \( \cot(z) \).

\[
|\cot(z)| = \left| \frac{\cos x \cosh y - i \sin x \sinh y}{\sin x \cosh y + i \cos x \sinh y} \right|
= \sqrt{\frac{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}}
\leq \sqrt{\frac{\cosh^2 y}{\sinh^2 y}}
= |\coth(y)|
\]

The hyperbolic cotangent, \( \coth(y) \), has a simple pole at \( y = 0 \) and tends to \( \pm 1 \) as \( y \to \pm \infty \).

Along the top and bottom of \( C_n \), \( (z = x \pm i(n + 1/2)) \), we bound the modulus of \( g(z) = \pi \cot(\pi z) \).

\[
|\pi \cot(\pi z)| \leq \pi |\coth(\pi(n + 1/2))|
\]
Along the left and right sides of $C_n$, ($z = \pm(n + 1/2) + iy$), the modulus of the function is bounded by a constant.

\[
|g(\pm(n + 1/2) + iy)| = \left| \frac{\cos(\pi(n + 1/2)) \cosh(\pi y) \mp i \sin(\pi(n + 1/2)) \sinh(\pi y)}{\sin(\pi(n + 1/2)) \cosh(\pi y) + i \cos(\pi(n + 1/2)) \sinh(\pi y)} \right|
\]

\[
= |\mp i\pi \tanh(\pi y)|
\]

\[
\leq \pi
\]

Thus the modulus of $\pi \cot(\pi z)$ can be bounded by a constant $M$ on $C_n$.

Let $f(z)$ be analytic except for isolated singularities. Consider the integral,

\[
\oint_{C_n} \pi \cot(\pi z) f(z) \, dz.
\]

We use the maximum modulus integral bound.

\[
\left| \oint_{C_n} \pi \cot(\pi z) f(z) \, dz \right| \leq (8n + 4)M \max_{z \in C_n} |f(z)|
\]

Note that if

\[
\lim_{|z| \to \infty} |zf(z)| = 0,
\]

then

\[
\lim_{n \to \infty} \oint_{C_n} \pi \cot(\pi z) f(z) \, dz = 0.
\]

This implies that the sum of all residues of $\pi \cot(\pi z) f(z)$ is zero. Suppose further that $f(z)$ is analytic at $z = n \in \mathbb{Z}$. The residues of $\pi \cot(\pi z) f(z)$ at $z = n$ are $f(n)$. This means

\[
\sum_{n=-\infty}^{\infty} f(n) = -\left( \text{sum of the residues of } \pi \cot(\pi z) f(z) \text{ at the poles of } f(z) \right).
\]
Result 13.10.1 If

$$\lim_{|z|\to\infty} |zf(z)| = 0,$$

then the sum of all the residues of $\pi \cot(\pi z)f(z)$ is zero. If in addition $f(z)$ is analytic at $z = n \in \mathbb{Z}$ then

$$\sum_{n=-\infty}^{\infty} f(n) = - (\text{sum of the residues of } \pi \cot(\pi z)f(z) \text{ at the poles of } f(z)).$$

Example 13.10.1 Consider the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2}, \quad a \notin \mathbb{Z}.$$

By Result 13.10.1 with $f(z) = 1/(z+a)^2$ we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = - \text{Res} \left( \pi \cot(\pi z) \frac{1}{(z+a)^2}, -a \right)$$

$$= -\pi \lim_{z \to -a} \frac{d}{dz} \cot(\pi z)$$

$$= -\pi \left( -\pi \sin^2(\pi z) - \pi \cos^2(\pi z) \right)$$

$$\frac{\sin^2(\pi z)}{\sin^2(\pi z)}.$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2(\pi a)}$$

Example 13.10.2 Derive $\frac{\pi}{4} = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots$. 

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Consider the integral
\[ I_n = \frac{1}{i2\pi} \int_{C_n} \frac{dw}{w(w - z)\sin w} \]
where \( C_n \) is the square with corners at \( w = (n + 1/2)(\pm 1 \pm i)\pi, \ n \in \mathbb{Z}^+ \). With the substitution \( w = x + iy \),
\[ |\sin w|^2 = \sin^2 x + \sinh^2 y, \]
we see that \( |1/\sin w| \leq 1 \) on \( C_n \). Thus \( I_n \to 0 \) as \( n \to \infty \). We use the residue theorem and take the limit \( n \to \infty \).

\[ 0 = \sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n\pi(n\pi - z)} + \frac{(-1)^n}{n\pi(n\pi + z)} \right] + \frac{1}{z\sin z} - \frac{1}{z^2} \]

\[ \frac{1}{\sin z} = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - z^2} \]
\[ = \frac{1}{z} - \sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n\pi - z} - \frac{(-1)^n}{n\pi + z} \right] \]

We substitute \( z = \pi/2 \) into the above expression to obtain
\[ \pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \cdots \]